Modeling by Stochastic Processes
Exercises 7

Problem 5 Let i and j be communicating states of a Markov chain. We want to show that i and j have the same period.

The case $d(i) = 0$ is trivial. Assume $d(i) \geq 1$ and $p_{ii}^{(s)} > 0$ for some $s > 0$. Then by definition $d(i)$ devides s. Since i and j communicate with each other, there exists m and n such that $p_{ij}^{(m)} > 0$ and $p_{ji}^{(n)} > 0$. So

$$p_{jj}^{(n+s+m)} \geq p_{ij}^{(n)} p_{ii}^{(s)} p_{ij}^{(m)} > 0.$$

Also, since $p_{ii}^{(2s)} > 0$, $p_{jj}^{(n+2s+m)} > 0$. So by definition, $d(j)$ devides $n+s+m$ and $n+2s+m$. Therefore $d(j)$ devides $(n+2s+m)-(n+s+m) = s$. Since s was arbitrary and $d(i)$ is by definition the greatest common divisor we get $d(j) \leq d(i)$. On the other hand, we can also argue in the same way w.r.t. the state j and obtain $d(i) \leq d(j)$. Alltogether this gives $d(i) = d(j)$.