1 Introduction

1.1 A view on the evaluation of risk

The role of mathematics

What skills should the modern actuary possess? How much is this influenced by the powerful modern computers? This book sees an actuary is a practitioner who is there to solve practical problems of risk in insurance and finance. Mathematics is an essential part and plays two different roles. One is as vendor of models that provide simplified descriptions of complicated risk processes. These models are usually stochastic. Examples are probability distributions of claim numbers and losses in general insuarnce, Markov processes describing how long we live or how often we become disabled in life insurance or interest rate and stock market fluctuations in finance.

Mathematics is from this point of view a *language*. It is a way risk is expressed, and it is a language we *must* be master. Otherwise statements of risk couldn't be related to the reality, it would be impossible to say what conclusions mean in any precise manner and nor could analyses be presented effectively to clients. Actuarial science is in this sense almost untouched by modern computational facilities. The basic concepts and models remain what they were, notwithstanding, of course, the strong growth of risk products throughout the last decades. This development may have had something to do with computers, but not much with computing per se.

But mathematics is also *deductions* where precise conclusions are derived from precise assumptions through the rules of logic. That is the way mathematics is taught at school and at university. It is here computing enters applied mathematical disciplines like actuarial science. More and more of these deductions are implemented in computers and carried out there. This has been going on for several decades. It leans on an endless growth in computing power, a true technological revolution that opens for simpler and more general computational methods which require less of users.

Risk methodology

The supreme example of such an all-purpose computational technique is **stochastic simulation**. Simplified versions of processes taking place in real life is then reproduced in the computer. Risk in finance and insurance is future, uncertain gains and losses, designated in this book by letters such as X and Y. Typical examples are compensations for claims in general insurance, pension schemes interrupted upon death in life insurance and future values of shares and bonds in finance. There are also secondary (or derived) products where values and pay-offs are channeled through certain contract clauses set up in advance. Such agreements are known as **derivatives** in finance and **re-insurance** in insurance.

The mathematical approach, today unanimously accepted, is through probabilities. Risks X and Y are then regarded as **random variables**. We shall know their values eventually (after the event), but for planning and control and to price risk-taking activities we need them in advance and must then fall back on their probabilities. This leads to a working process as the one depicted in Figure 1.1. The real world on the left is an enormously complicated mechanism (denoted M) that yields a future X. We shall never know M, though our paradigm is that it does exist as a well-defined, stochastic mechanism. Since it is beyond reach, a simplified version $\widehat{M}$ in constructed in its place

The world	*In the computer*	*Sources for* $\widehat{M}$
of risk		*Historical experience*
$M \to X$	$\widehat{M} \to X^*$	*The implied market view*
	$\uparrow$	*Deductions from no-arbitrage*
	Assumed mechanism	*Judgement, physical modelling*

Figure 1.1 *The working process: Steps when evaluating a risk* X.

and used to draw conclusions about X. We are rarely trying to predict what X is going to be. Usually our interest is its region of variation, for example its expected value (used for valuation) or its percentiles (used for control). Note that everything falls apart if $\widehat{M}$ deviates too strongly from the true mechanism M. This issue of **error** is a very serious one indeed. Chapter 7 is an introduction.

What there is to go when $\widehat{M}$ is erected is listed on the right in Figure 1.1. Learning from the past is an obvious source (but not all of it is relevant). In finance current asset prices relate market opinion on the future. This so-called **implied view** is briefly introduced in Section 1.4, and there will be more in part III. Then there is the theory of **arbitrage** where risk-less financial income is regarded as impossible. This innocently looking no-arbitrage condition has wide implications which is discussed in Chapter 14. In practice there might also be some personal judgement behind $\widehat{M}$, but this is not suitable for general argument, and nor shall we go into physical modelling used in large claims insurance where physical damages from hurricanes, earthquakes or floods are imitated in the computer. This book is about how $\widehat{M}$ is constructed from the first three sources (historical data above all), how it is implemented in the computer and how the computer model is used to determine the probability distribution of X.

The computer model

The real risk variable X will materialize only once. The economic result of an financial investment in a particular year is an unique event as is the aggregated claim against an insurance portfolio during a certain period of time. With the computer model that is different. Once it has been set it up it can be played as many times as we please. Let $X_1^*, \dots X_m^*$ of be m realizations of X. They tell us which values of X are likely and which are not and how badly things might become if we are unlucky. The *-marking will be used throughout to distinguish computer simulations from real variables and m will always denote the number of simulations.

The method portrayed on the left in Figure 1.1 is known as the **Monte Carlo** method or as **stochastic simulation**. It belongs to the realm of numerical integration; see Evans and Schwarz (2000) for a summary of this important branch of numerical mathematics. Monte Carlo integration dates far back. It is computationally slow, but other numerical methods (that might do the job faster) often require more expertise to use and bog down for high-dimensional integrals which is precisely what is often needed in practice. The Monte Carlo method is unique in handling many variables well.

What is the significance of numerical speed anyway? Does it really matter that some specialized technique (demanding more time and know-how to implement) is (say) one hundred times faster when the one we use only takes a second? If the procedure for some reason is to be repeated in a loop thousands of times, it *would* matter. Often, however, slow Monte Carlo is quite enough, and, indeed, the practical limit to its use is moving steadily as computers become more and more powerful. How far have we got? The personal computer on the author's desk (not particularly advanced) could produce (2004) around 3 million drawings from the Pareto distribution per second (using Algorithm 2.9, implemented in Fortran) and a similar number from the normal (Algorithm 2.1). That is 1000 claims in an insurance portfolio simulated 10000 times (i.e. 10 *million* draws) completed in about three seconds!

One of the aims of this book is to demonstrate how these opportunities are utilized. Principal issues are how simulations programs are designed, how they are modified to deal with related (but different) problems and how different programs are merged to handle situations of increasing complexity with several risk factors contributing jointly. The versatility and usefulness of Monte Carlo is indicated in Section 1.5 (and in Chapter 3 too). By mastering it you are well equipped to deal with most of what that comes up and are not stuck when software packages lack what you need. What platform should you go for? Algorithms in this book have been written in the pseudo-code of Algorithm 1.1 that goes with everything. Excell and Visual Basic are a standard in the industry and may be used even for simulation. Much higher speed is obtained with C, Pascal or Fortran, and people are in the opinion of this author well adviced to learn software like those. There are other possibilities as well. Much can be achieved with a platform you know!

How the book is planned

This is an elementary treatise, at least in terms of the mathematics used. No more than a bare necessity of ordinary and probabilistic calculus is demanded, and mathematical arguments rarely exceed calculations of means and variances. Models are whenever possible presented the way they are simulated in the computer. Their probabilistic description in terms of distributions is often complicated stuff, but this is required only when when models are fitted by maximum likelihood, and 'advanced' modelling can therefore be reached quickly. Nor is it necessary to rely on artillery as heavy as stochastic analysis and Ito integration to understand what modern financial derivatives are about (see Chapter 14). The matematics in this book is always in discrete time with modelling in continuous time as limits when the time incement (denoted h) approaches 0.

The next three sections introduce main concepts of risk in insurance and finance with the core of the mathematical notation needed. Why stochastic simulation is such a unifying tool will be indicated in Section 1.5. This is a forerunner of the entire Part I where the Monte Carlo method and its potential is presented jointly with elementary models in insurance and finance. The idea is to get Monte Carlo settled early as a vehicle for analysis, learning and communication. Other major tools for risk studies are stochastic modelling of dependence (Chapter 5 and 6) and historical estimation and error (Chapter 7). Part I carries mathematical, statistical and computational methods from one application area to another with examples from everywhere. The two other parts of the book deal with general insurance (Part II) and life insurance and financial risk (Part III). The treatment is now more systematic with more complex models and situations being examined.

1.2 Insurance risk: Basic concepts

Introduction

Property or **general** insurance is economic responsibility for unexpected events such as fires or accidents passed on (entirely or in part) to an insurer against a fee. The contract, known as a **policy**, releases compensation (known as a **claim**) according to certain clauses. A central quantity is the total claim X amassed during a certain period of time (typically a year). Often $X = 0$ (no events), but on rare occasions X is huge. An insurance company (if properly run) copes whatever happens. It has a **portfolio** of many such risks and only a few of of them materialize. The control of the total uncertainty involved is a major theme in general insurance.

Life insurance is also built up from random payments X. **Term** insurance where beneficiaries receive compensation upon the death of the the policy holder is similar to property insurance in that unexpected events lead to pay-offs. **Pension schemes** are the opposite. Now the payments go on as long as the insured is alive, and they are likely, not rare. Yet the basic approach remains the same with random variables X expressing the uncertainty involved.

Pricing of insurance risk

Transfers of risk through X do not take place for free. The fee (or **premium**), always charged in advance, depends on the market conditions, but the expectation is a guideline. Introduce

$$\pi^{\text{pu}} = E(X), \tag{1.1}$$

which is known as the **pure** premium and defines a break-even situation. A company receiving π^{pu} for its services will in the absence of all overhead cost and all financial income neither earn nor lose in the long run. This is a consequence of the law of large numbers in probability theory; see Appendix A.

Such a pricing strategy is (of course) out of the question, and companies add **loadings** γ on top of π^{pu}. The premium charged is then

$$\pi = (1 + \gamma)\pi^{\text{pu}}, \tag{1.2}$$

and we may regard $\gamma\pi^{\text{pu}}$ as the cost of risk. It is thoroughly influenced by the market situation, and are in many branches of insurance known to exhibit strong fluctuations; see Section 11.5 for a simple model. There has been attempts to determine γ from theoretical arguments, see Young (2004) is a good review, but these efforts are not much used in practice and will not be considered.

The significance of the loading concept is the insulation of market impact from the insurance process itself. Another issue is whether the pure premium really is known. When stochastic models for X are introduced in later chapters, it will emerge that there are always unknown quantities (parameters, probability distributions) determined from experience or even assessed informally if hard historical data are lacking. This creates a crucial distinction between the true π^{pu} with perfect knowledge of the underlying situation and the one $\hat{\pi}^{\text{pu}}$ used for analysis and decisions. The discrepancy between what we seek and what we get is a fundamental issue of **error** that is present everywhere (see Figure 1.1), and there is special notation for it. A parameter or quantity with a $\hat{}$ such as $\hat{\psi}$ means an estimate or assessment of an underlying, unknown ψ. Chapter 7 offers a general discussion of errors and how they are confronted.

Portfolios and solvency

A second major theme in insurance is **control**. Companies are obliged to to set aside funds to cover furure obligations. Indeed, this is a major theme in the *legal* definition of insurance. An insurance company carries responsibiity for many polices. It will lose on some and gain on others. In property insurance policies without accidents are profitable, those with large claims are not. Long lives in pension schemes lead to losses, short ones to gains. *On portfolio level gains and losses average out.* This is the beauty of a large agent handling many risks simultaneously.

Suppose the portfolio consists of J policies with claims $X_1, \ldots, X_J$. The total claim against the portfolio is then

$$\mathcal{X} = X_1 + \ldots + X_J \tag{1.3}$$

where caligraphical letters like $\mathcal{X}$ will be used for quantities applying to portfolios. We are certainely interested in $E(\mathcal{X})$, but equally important is its distribution. Regulators demand sufficient funds to cover $\mathcal{X}$ with high probability. The mathematical formulation is in terms of a percentile q_ϵ which is the solution of the equation

$$\Pr(\mathcal{X} > q_\epsilon) = \epsilon \tag{1.4}$$

where ϵ is a small number (for example 1%). The amount q_ϵ is known as the **solvency capital** or the **reserve**. Percentiles are used in finance too and is then often called Value at Risk (or **VaR** for short). As elsewhere the true q_ϵ we seek is not same as the estimated $\hat{q}_\epsilon$ we get; see Chapter 7.

Risk ceding and re-insurance

Risk is ceded from ordinary policy holders to companies, but companies do the same thing between themselves. This is known as **re-insurance**, and the ceding company is called the **cedent**. The rationale *could* be the same; i.e. that a financially weaker agent is passing risk to a stronger one. In reality even the largest of companies do this to diversify risk, and financially the cedent may be as strong as the re-insurer. There is now a chain of responsibilities that can be depicted as follows:

$$\begin{array}{ccccc} \text{original clients} & \longrightarrow & \text{cedent} & \longrightarrow & \text{re-insurer.} \\ \mathcal{X}\ (\textit{primary}) & & \mathcal{X}^{\text{ce}} = \mathcal{X} - \mathcal{X}^{\text{re}} & & \mathcal{X}^{\text{re}}\ (\textit{derived}) \end{array}$$

The original risk $\mathcal{X}$ is split between cedent and the re-insurer through two separate relationships where the cedent part $\mathcal{X}^{\text{ce}}$ is net and the difference between two cash flows. Of course $\mathcal{X}^{\text{re}} \leq \mathcal{X}$; i.e. the responsibility of the re-insurer is always *less* than the original claims. Note the caligraphic style that applies to portfolios. There may in practice be several rounds of such cedings in complicated networks extending around the globe. One re-insurer may go to a second re-insurer and so on. Modern methods provide the means to analyse risk taken by an agent who is far away from the primary source. Ceding and re-insurance are tools used by managers to tune portfolios to a desired risk profile.

1.3 Financial risk: Basic concepts

Introduction

Gone are the days where insurance liabilities were handled insulated from assets and where insurance companies carried all financial risk themselves. One trend is ceding to customers. In contries like the US and Britain insurance products with financial risk integrated have been sold for decades under names such as unit link or universal life. The rationale is that clients receive higher financial income in expectation in exchange for carrying more risk. Pension plans are today increasingly contributed benfits (or **CB**) where financial risk rests with the individual members. There is also much interest in investment strategies taylored to liabilities, in particular how they distribute over time. That is known as asset liability management (**ALM** for short) and is discussed in Chapter 15. The present section and the next one reviews the main concepts of finance.

Rates of interest
An ordinary bank deposit v_0 grows to $(1+r)v_0$ at the end of one period and to $(1+r)^K v_0$ after K periods. Here r, the **rate of interest**, depends on the length of the period. Suppose interest is compounded over K segments, each of length $1/K$ so that the total time is one. With interest per segment being r/K the value of the account becomes

$$(1+\frac{r}{K})^K v_0 \rightarrow e^r v_0, \qquad \text{as} \qquad K \rightarrow \infty,$$

after one of the most famous limits of mathematics. Interest earnings may therefore be cited as

$$rv_0 \qquad \text{or} \qquad (e^r - 1)v_0,$$

depending on whether we include 'interest on interest'. The second form implies continuous compounding of interest and higher earnings ($e^r - 1 > r$ if $r > 0$), and now $(e^r)^k = e^{rk}$ takes over from $(1+r)^k$. It doesn't really matter which form we choose, since they can be made equivalent by adjusting r.

Financial returns
Let V_0 be the value of a financial asset at the start of a period and V_1 the value at the end of it. The relative gain

$$R = \frac{V_1 - V_0}{V_0}, \tag{1.5}$$

is called the **return** of the asset. Solving for V_1 yields

$$V_1 = (1+R)V_0, \tag{1.6}$$

which shows that RV_0 is financial income. Clearly R acts like interest, but it is more than that. Interest is a fixed benefit offered by a bank (or an issuer of a very secure bond) in return for making a deposit and is risk-free. Shares of company stock, on the other hand, are fraught with risk. They may go up (R positive) or down (R negative). When dealing with such assets, V_1 (and hence R) is determined by the market whereas with ordinary interest r is given and V_1 follows.

The return R is the more general concept and is a random variable with a probability distribution. Take the randomness away, and we are back to a fixed rate of interest r. As r depends on the time between V_0 and V_1, so does the distribution of R; how will appear many times in this book. Whether the rate of interest r really *is* risk-free is not so obvious as it seems. True, you do get a fixed share of your deposit as a reward, but that does not tell its worth in **real** terms when price increases are taken into account. Indeed, over longer time horizones risk due to inflation may be huge and reduce the real value of cash deposits and bonds. Saving money at a bank at a fixed rate of interest also may also bring **opportunity cost** if the market rate after a while overturns what you get. These issues are discussed and integrated with other sources of risk in Part III.

Log-returns

Economics and finance have often constructed stochastic models in terms of R directly. An alternative is the **log-return**

$$L = \log(1 + R), \tag{1.7}$$

which by (1.5) can be written $L = \log(V_1) - \log(V_0)$; i.e. as a difference on logarithmic scale. The modern theory of financial derivatives (Section 3.5 and Chapter 14) is based on L. Actually L and R do not necessarily deviate that strongly since the Taylor series of $\log(1 + R)$ yields

$$L = R - \frac{R^2}{2} + \frac{R^3}{3} + \dots,$$

where R (a relatively small number) dominate so that $L \doteq R$, at least over over short periods. The distributions of R and L must then be fairly similar too; see Section 2.3. However, this is not to say that the discrepancy is unimportant. It depends on the amount of random variation present, and the longer the time horizon the more L deviate from R. There are calculations in Section 5.4.

Financial portfolios

Investments are often spread on many assets as **baskets** or financial **portfolios**. By intuition this must reduce risk; see Section 5.3 where the issue is discussed. A central quantity is the portfolio return, denoted $\mathcal{R}$ (in caligrahic style). Its relationship to the individual returns R_j of the assets is as follows. Let $V_{10}, \dots, V_{J0}$ be investments in J assets. The portfolio value is then

$$\mathcal{V}_0 = \sum_{j=1}^{J} V_{j0} \qquad \text{growing at the end of the period to} \qquad \mathcal{V}_1 = \sum_{j=1}^{J}(1 + R_j)V_{j0}.$$

Subtract $\mathcal{V}_0$ from $\mathcal{V}_1$ and divide on $\mathcal{V}_0$, and you get the portfolio return

$$\mathcal{R} = \sum_{j=1}^{J} w_j R_j \qquad \text{where} \qquad w_j = \frac{V_{0j}}{\mathcal{V}_0}. \tag{1.8}$$

Here w_j is the **weight** on asset j. Note that

$$w_1 + \dots + w_J = 1. \tag{1.9}$$

Financial weights define how the portfolio distributes on individual assets and will in this book always satisfy this normalizing condition.

The mathematics allow negative w_j. With bank deposits this corresponds to borrowing. It is also possible with shares and is then known as **short selling**. A loss due to a negative development is then carried by somebody else. The mechanism is as follows. Our short contract with a buyer is to sell shares at the end of the period at an agreed price. At that point we shall have to buy at market price, gaining if it is lower than our agreement, losing if not. Short contracts may be an instrument to lower risk (see Section 5.3) and requires **liquidity**; i.e. assets that are traded regularly.

1.4 Financial risk over time

Introduction
A huge number of problems in finance (and in insurance too) have time as a central ingredient, and this requires additional quantities and concepts and a rather elaborate mathematical notation. Time will be run on equidistant sequences of the form

$$t_k = kh, \qquad k = 0, 1, \ldots, K, \tag{1.10}$$

where $h > 0$ is the time increment and k the time index. Variables such as interest rates r_k, returns R_k, portfolio values $\mathcal{V}_k$ and insurance liabilities $\mathcal{X}_k$ are all assigned values at t_k. We observe them at $t_0 = 0$ (in this book always the present), but what they are going to be in the future ($k > 0$) is unknown although stochastic models suggest which values are likely and which are not. The past will sometimes be indexed by negative k.

Many different time scales h are required. Accountancy is typically annual, and both years, quarters and months may be appropriate for insurance liabilities and financial variables when followed over decades. There is also scope for much shorter time increments, Indeed, many model constructions are based on infitisemal h; i.e. we let $h \to 0$! This is known as **continuous** time and is a trick to find simple mathematical solutions. Parameters are then often cited as **intensities** which are quantities per time unit. A case in point is interest rate being denoted rh rather than r. Claim frequencies in property insurance (Chapter 8) and mortalities in life insurance (Chapter 12) are other prominent examples of intensities.

K-step quantities
Let v_0 be the value of a financial asset earning returns $R_1, \ldots, R_K$ in the periods ahead. By $t_K = Kh$ it is worth

$$V_K = (1 + R_1)(1 + R_2) \cdots (1 + R_K)V_0 = (1 + R_{0:K})v_0,$$

which defines $R_{0:K}$ as a **K-step** return over K periods through

$$\underbrace{1 + R_{0:K} = (1 + R_1) \cdots (1 + R_K)}_{\textit{ordinary returns}} \qquad \text{and also} \qquad \underbrace{L_{0:K} = L_1 + \ldots + L_K,}_{\textit{log-returns}} \tag{1.11}$$

where $L_k = \log(1 + R_k)$ and $L_{0:K} = \log(1 + R_{0:K})$. The sum on the right is the logarithm of the product on the left.

Interest rates is a special case and an important one. If $r_1, \ldots, r_K$ are future rates, then the rate of interest from t_0 to t_K is

$$r_{0:K} = (1 + r_1)(1 + r_2) \cdots (1 + r_K) - 1. \tag{1.12}$$

This reduces to $r_{0:K} = (1+r)^K - 1$ if all $r_k = r$, but in practice r_k will float in a way that is unknown at $t_0 = 0$.

Forward rates of interest
Future interest rates like $r_{0:K}$ is crucial, yet hopeless to predict from mathematical models (you

see why in Section 6.4), but there is also a market view that conveys the so-called **implied** rates. Consider an asset of value v_0 that will be traded at time t_K for a price $V_0(K)$ agreed today. Such contracts are called **forwards** and define inherent rates of interest $r_0(0{:}K)$ through

$$V_0(K) = \{1 + r_0(0{:}K)\}v_0 \qquad \text{or} \qquad r_0(0{:}K) = \frac{V_0(K)}{v_0} - 1. \tag{1.13}$$

Note the difference from (1.12) where $r_{0:K}$ is uncertain whereas now $V_0(K)$ and hence $r_0(0{:}K)$ is fixed by the contract and known at $t_0 = 0$. Forward rates can in practice be deduced from many different sources, and the results are virtually identical. If not, there would have been a market inconsistency which would have opened for money earning schemes with no risk attached (more on this in Chapter 14).

We are often interested in breaking the rate $r_0(0{:}K)$ down on its average value $\bar{r}_0(0{:}K)$ per period. The natural definition is

$$1 + r_0(0{:}K) = \{1 + \bar{r}_0(0{:}k)\}^K \quad \text{which yields} \quad \bar{r}_0(0{:}K) = \{1 + r_0(0{:}K)\}^{1/K} - 1, \tag{1.14}$$

and as K is varied, the sequence $\bar{r}_0(0:K)$ traces out the **interest rate curve** or **yield curve**, published daily in the financial press.

Present and fair values

What is the value today of receiving B_1 at time t_1? Surely it must be $B_1/(1 + r)$ which grows to precisely B_1 when interest is added. More generally, B_k at t_k is under constant rate of interest worth $B_k/(1+r)^k$ today. This motivates the **present value** (PV) as the value of a payment stream $B_0, \ldots, B_K$; i.e.

$$\text{PV} = \sum_{k=0}^{K} d_k B_k \qquad \text{where} \qquad d_k = \frac{1}{(1+r)^k}, \tag{1.15}$$

which is a popular criterion in all spheres of economic life. It applies even when $B_0, \ldots, B_K$ are stochastic (the present value is then stochastic as well). Individual payments may be both positive and negative.

The quantities $d_k = 1/(1 + r)^k$ (or $d_k = e^{-rk}$ if continuous compounding of interest is used) are known as **discount factors**; they devaluate or discount future income. In life insurance r is called the **technical rate**. The value to use isn't obvious, especially not with payment streams decades ahead. It isn't easy to know what interest rates will become over time spans like those! Market discounting is an alternative. The coefficient d_k in (1.15) is then replaced by

$$d_k = \frac{1}{\{1 + \bar{r}_0(0{:}k)\}^k} = \frac{1}{1 + r_0(0{:}k)} \qquad \text{or} \qquad d_k = P_0(0{:}k), \tag{1.16}$$

where $P_0(0:k)$ comes from the bond market; see below. Instead of choosing the technical rate r administratively, we use the market view. The resulting valuation is known as **fair value** and holds obvious attraction. A disadvantage is that the discount sequence fluctuates up and down with the market, and this brings considerable uncertainty (Section 15.4) even if there was none in the beginning.

Bonds and yields

Governments and private companies raise capital by issuing **bonds**. In return for money received up-front the issuer makes *fixed* payments at pre-determined points in time t_k, $k = 0, 1, \ldots K$. The end transfer (known as the **face** of the bond) is a big one, and the earlier payments can be seen as interest on a loan that size, but it is simplest to define a bond as a fixed cash flow. How long it lasts varies enormously, from a year or less to up to half a century or even more! Bonds have a huge second-hand market and are traded regularly.

Should bonds be valued through present or fair values? Actually it is the other way around. The present value is *given* by what the market is willing to pay, and the rate of interest determined by the resulting equation. We are dealing with a fixed payments stream $B_0, \ldots, B_K$. Let v_0 be its value at $t_0 = 0$. The **yield** y from buying the rights to the stream is then the solution of the equation

$$v_0 = \sum_{k=0}^{K} \frac{B_k}{(1+y)^k}. \tag{1.17}$$

With more than one payment a numerical method is needed to determine y; see Section C.4.

A special case is the **zero-coupon** bond or **T-bond** for which $B_0 = \ldots B_{K-1} = 0$. It is **unit-faced** if $B_K = 1$. Now the only transaction occurs at maturity t_K, and in a market operating rationally its yield y is the same as the forward rate of interest $\bar{r}_0(0\!:\!K)$. The price of unit-faced T-bonds will be denoted $P_0(0\!:\!K)$. This is what is charged today for the right to receive one money unit at t_K and relates to the forward rate of interest through

$$P_0(0\!:\!K) = \frac{1}{1 + r_0(0\!:\!K)} = \frac{1}{\{1 + \bar{r}_0(0\!:\!K)\}^K} \tag{1.18}$$

which is again obvious since anything else brings risk-less financial income. The prices $P_0(0\!:\!K)$ will be used a lot in Chapters 14 and 15.

Duration

The timing of bonds and other fixed payment streams is often measured through their **duration** $\mathcal{D}$. There are several versions which vary in detail. The one used here is

$$\mathcal{D} = \sum_{k=0}^{K} q_k t_k \qquad \text{where} \qquad q_k = \frac{B_k(1+r)^{-k}}{\sum_{i=0}^{K} B_i (1+r)^{-i}}. \tag{1.19}$$

Note that the sequence $q_0, \ldots, q_K$ is a probability distribution (it adds to one) with q_k being proportional to the present value of the k'th payment. This means that the duration $\mathcal{D}$ expresses how long the cash flow $B_0, \ldots, B_K$ lasts 'on average'.

For a zero-coupon bond maturing at $t_K = Kh$, we have

$$q_K = 1 \qquad \text{and} \qquad q_k = 0, \ \text{for} \ k < K.$$

so that $\mathcal{D} = t_K$, a sensible result! A bond with fixed coupon payments and a final (much larger) face has duration between $t_K/2$ and t_K.

Investment strategies

Long term management of financial risk is usually concerned with different classes of assets which fluctuate jointly. Let $\mathcal{R}_k$ be the portfolio return in period k. The account $\{\mathcal{V}_k\}$ then evolves according to

$$\mathcal{V}_k = (1 + \mathcal{R}_k)\mathcal{V}_{k-1}, \qquad k = 1, 2, \ldots, \tag{1.20}$$

where the link of $\mathcal{R}_k$ to the individual assets is through (1.8) as before. If R_{jk} is the return of asset j in period k, then

$$\mathcal{R}_k = \sum_{j=1}^{J} w_j R_{jk}. \tag{1.21}$$

The weights $w_1, \ldots, w_J$ define different investment strategies. One way is to keep them fixed, as in (1.21) where they are the same for all k. This is not achieved automatically since individual investments develope unequally so that their relative values change. Weights can only be kept fixed by buying assets that have gone badly and selling those that have been successful. Restructuring financial weights in such a way is known as **rebalancing**.

An alternative line is to allow weights to float freely. Mathematically this is more conveniently expressed through

$$\mathcal{V}_k = \sum_{j=1}^{J} V_{kj}, \qquad \text{where} \qquad V_{kj} = (1 + R_{kj})V_{k-1,j}, \quad j = 1, \ldots, J,$$

and the emphasis is on the assets rather than on their returns. There is more on investments strategies in Chapter 15.

1.5 Method: A unified beginning

Introduction

How to make the preceding quantities and concepts flourish? Stochastic models and Monte Carlo are needed! The following simple example introduces both. Consider the recursion

$$Y_k = a_k Y_{k-1} + X_k, \quad k = 1, 2 \ldots, \qquad \text{starting at} \qquad Y_0 = y_0 \tag{1.22}$$

where $X_1, X_2, \ldots$ are independent random variables acting as **drivers** of the output $Y_1, Y_2, \ldots$, in shorthand notation $\{X_k\}$ and $\{Y_k\}$. Many important situations are covered, as will emerge below. The second series $\{a_k\}$ may be fixed coefficients, but another possibility is $a_k = 1 + r$ where r is a rate of interest. Now $\{Y_k\}$ are values of an account influenced by random input. A more advanced version is

$$\underset{\textit{financial risk}}{a_k = 1 + \mathcal{R}_k} \qquad \text{and} \qquad \underset{\textit{insurance risk}}{X_k = -\mathcal{X}_k,} \tag{1.23}$$

and two different sources of risk that might themselves demand extensive modelling and simulation are integrated; see Section 15.6. Here the target is a more modest one. A simple Monte Carlo algorithm and notation for such schemes will first be presented and then four simple examples. The aim is to introduce a general line of attack and indicate the power of Monte Carlo for problem solving, learning and communication.

Monte Carlo algorithms and notation

Let $Y_1, \dots, Y_K$ be the first K variables of the sequence (1.22). How they are simulated is indicated by the following scheme (a skeleton!) which defines the first algorithm of the book:

Algorithm 1.1 Basic recursion

0 Input: y_0, $\{a_k\}$.
1 $Y_0^* \leftarrow y_0$ *%Initialisation*
%Draw K here if random
2 For $k = 1, \dots, K$ do
3 Sample X_k^* *%Many possibilities*
4 $Y_k^* \leftarrow a_k Y_{k-1}^* + X_k^*$ *%New value*

5. Return $Y_0^*, \dots Y_K^*$ (or just Y_K^*)

After initialisation (on Line 1), the random terms X_k^* are drawn (Line 3) and the the preceding values Y_{k-1}^* revised. All simulated variables are $*$-marked, a convention that will be followed everywhere. The backward arrow $\leftarrow$ signifies that the variable on the left is assigned the value on the right. It is a more convenient notation than an ordinary equality sign. For example, when only the *last* value Y_K^* is wanted (as is frequent), statements like $Y^* \leftarrow aY^* + X^*$ simply overwrite Y^*, and values of the past are *not* stored in the computer. The % symbol will be used to insert comments.

A huge number of simulation experiments in insurance and finance fit this scheme or some simple variation of it which suggests a fairly stable pattern for Monte Carlo programming that can be lifted from one problem to another. Is K in Algorithm 1.1 random (as in Example 2 below)? Draw it prior to entering the loop on Line 2. Random a_k as in (1.23)? Similar, remove it from the input list and generate it before updating Y_k^* on Line 4.

Example 1: Term insurance

Consider J contracts where the death of policy holders release one-time payments. The likelihood of this depends on age and sex, and the probabilities would be available on file. This situation is so simple that it is possible to examine the portifolio liability and its uncertainty through mathematics (it is done in Section 3.4), but the point now is to use Monte Carlo. Let X_j be the pay-off for policy j, either 0 when the policy holder survives or s_j when he does not. The stochastic model is

$$\Pr(X_j = 0) = p_j \qquad \text{and} \qquad \Pr(X_j = s_j) = 1 - p_j,$$

where p_j is the chance of survival. A simulation goes through the entire portfolio, reads policy information from file, draws those who die (whom we have to pay for) and adds all the payments together. In Algorithm 1.1 take $K = J$, $a_k = 1$ and X_j^* is either 0 or s_j; details in Section 3.4.

The example in Figure 1.2 shows annual expenses for $J = 10000$ policies for which $s_j = 1$ for all j (money unit: one million US$). All policy holders were between 30 and 60 years of age. Survival probabilities and the age distribution are as specified in Section 3.4. One hundred parallel runs

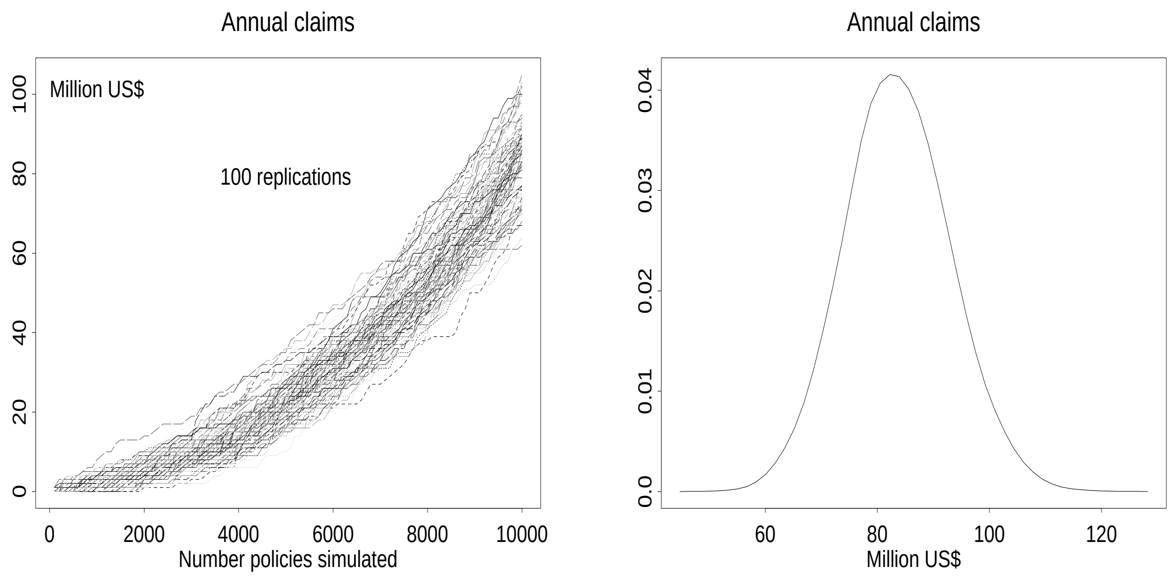

Figure 1.2 *Simulations of term insurance. Left:* 100 *parallel runs through insurance portfolio. Right: Annual density function obtained from* $m = 10000$ *simulations.*

through the portfolio are plotted jointly on the left showing how the simulations evolve. The curved shape has no significance. It is due to the age of the policy holders having been ordered on the file so that the young ones with low death rates are examined first. What counts is the variation at the end which is between 65 and 105 million. Another picture is provided by the probability density function on the right which has been estimated from the simulations by means of the kernel method of Section 2.2 (larger experiment needed). The Gaussian shape follows from the Lindeberg extension of the central limit theorem (Appendix A.4). In life insurance such risk is often ignored, but in this example uncertainty isn't negligible.

Example 2: Property insurance
A classic model in property insurance is identical risks. Claims are then (on average) equally frequent for everybody, and losses exhibit no systematic variation between individuals. The portfolio payout then becomes

$$\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}$$

where $\mathcal{N}$ is the number of insurance incidents and $Z_1, Z_2, \ldots$ their cost. In Algorithm 1.1 $K = \mathcal{N}$ (and drawn prior to the loop), $a_k = 1$ and $X_k = Z_k$; for details see Algorithm 3.1 in Chapter 3.

The example in Figure 1.3 was run with annual claim frequency 1% per policy and Poisson distributons for $\mathcal{N}$. Losses $Z_1, Z_2, \ldots$ were drawn from the empirical distribution function (Section 9.2) of the Danish fire data introduced in Section 9.6. The latter is a record of more than 2000 industrial fires, the largest going up to several hundred million Danish kroner (divide on seven/eight for euros). Figure 1.3 shows the density function for the portfolio liabilities for a 'small' portfolio ($J = 1000$) on the left and a 'large' one ($J = 100000$) on the right. The uncertainty is now greater than in the preceding example. For the small portfolio on the left the density function is strongly

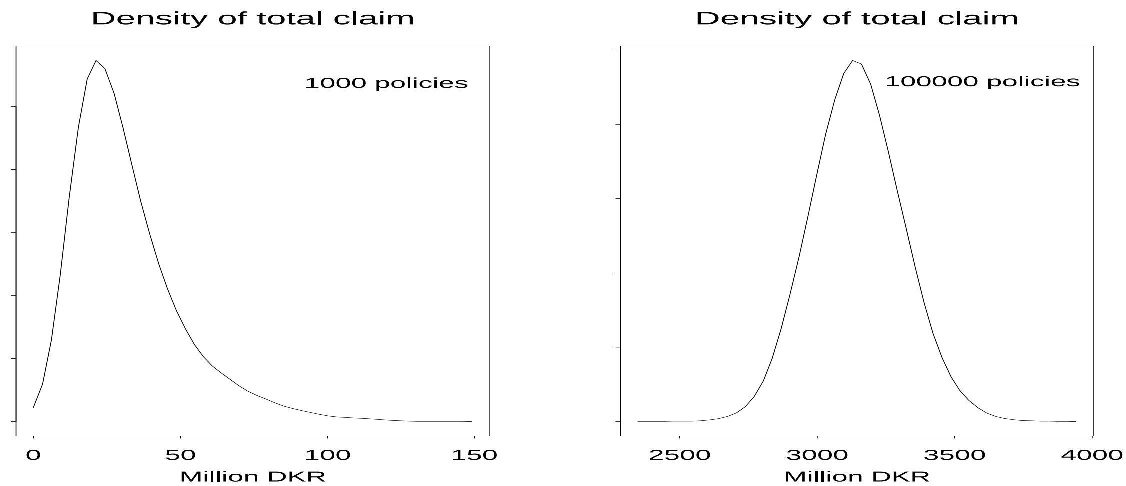

Figure 1.3 *Density functions of the total claim against portfolio of fire risks (seven/eight Danish kroner (DKK) for one euro).*

skewed towards the right, but as the portfolio grows, this asymmetry is straightened out, and the distribution becomes more Gaussian. The central limit theorem tells us it must be so.

Example 3: Reversion to mean
Monte Carlo is useful to examine the behaviour of stochastic models. Consider interest rates, equity volatilities, rates of inflation and exchange rates which all tend to fluctuate between certain, not clearly defined limits. If they move too far out on either side, there are forces in the economy that pull them back again. This is known as **reversion to mean** and applies to many (but not all) financial variables.

Interest rates is an important example. One of the most popular models, proposed by Vasiček (1977), is the recursion

$$r_k = Y_k + \xi \qquad \text{where} \qquad Y_k = aY_{k-1} + \sigma\varepsilon_k, \quad k = 1, 2, \ldots, \tag{1.24}$$

starting at $Y_0 = r_0 - \xi$. Here ξ, a and σ are fixed parameters and $\{\varepsilon_k\}$ independent and identically distributed variables with mean 0 and standard deviation 1. The model is known as an auto-regression of order one and is examined in Section 5.6. Here the objective is simulation which is carried out by taking $a_k = a$ in Algorithm 1.1 and adding ξ to the output $Y_1^*, Y_2^* \ldots$ so that Monte Carlo interest rates $r_1^*, r_2^*, \ldots$ are produced.

Figure 1.4 shows Monte Carlo scenarios under the following two models:

$$\underbrace{r_0 = 3\%,\ \xi = 7\%,\ a = 0.70,\ \sigma = 0.016}_{\text{'rapid' change}} \qquad \underbrace{r_0 = 3\%,\ \xi = 7\%,\ a = 0.95,\ \sigma = 0.007.}_{\text{'slow' change}}$$

The time scale is annual, and the series $\{\varepsilon_k\}$ in (1.24) is Gaussian. Note that the simulations start at $r_0 = 3\%$, much lower than the long-term average ($\xi = 7\%$) around which the interest rate eventually fluctuates. That level is quickly reached with the 'rapid' model scenario in Figure 1.4

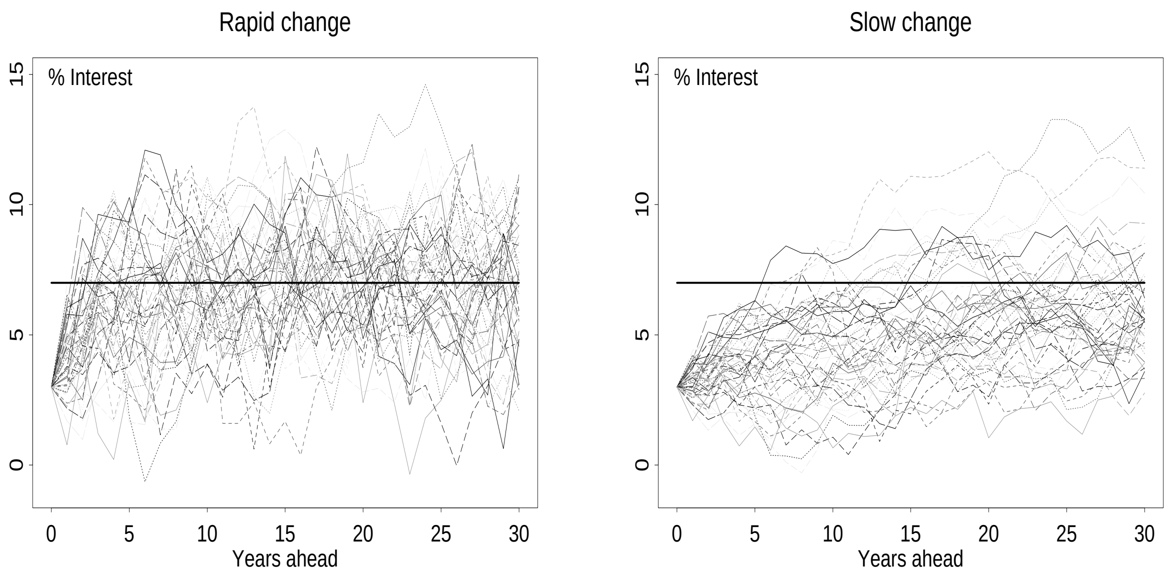

Figure 1.4 *Simulations of the annual rate of interest from the Vasiček model.*

left, and after about five years there is no *systematic* change in patterns. Such phenomena are called **stationary**, and they are discussed in Section 5.6. The same thing is observed with the other, 'slow' scenario on the right, but it now takes much longer (because the coefficient a is closer one), and after 25 years movements may still be slightly on the rise upwards (*not* realistic in practice). Stationarity requires that $-1 < a < 1$, and the behaviour of the model changes completely outside this interval as we shall see next.

Example 4: Equity over time
Stock prices $\{S_k\}$ are *not* mean reverting. They are (unlike interest rates and inflation) traded commodities, and had it been possible to identify systematic factors that drove them up and down, we would be able to act upon them and earn money. But that opportunity would be available to everybody, removing the forces we were utilizing and rendering the idea useless. Models for equity are therefore very different from those for interest rates, and based on stochastically independent returns. A common specification is

$$R_k = \exp(\xi + \sigma\varepsilon_k) - 1, \qquad k = 1, 2\ldots, \tag{1.25}$$

where ε_k is a sequence of independent random variables with mean zero and standard deviation one. By definition $S_k = (1 + R_k)S_{k-1}$ so that

$$S_k = \exp(\xi + \sigma\varepsilon_k)S_{k-1}, \qquad k = 1, 2\ldots, \qquad \text{starting at} \qquad S_0 = s_0. \tag{1.26}$$

This is known as a **geometric random walk**, and $\log(S_k)$ follows an ordinary random walk; see Section 5.5 where such models are discussed.

Their behaviour can be studied through Monte Carlo using Algorithm 1.1. Apply it to $Y_k = \log(S_k)$ (using $a_k = 1$ and $X_k = \xi + \sigma\varepsilon_k$) and convert the simulations by taking $S_k^* = e^{Y_k^*}$. The simulated

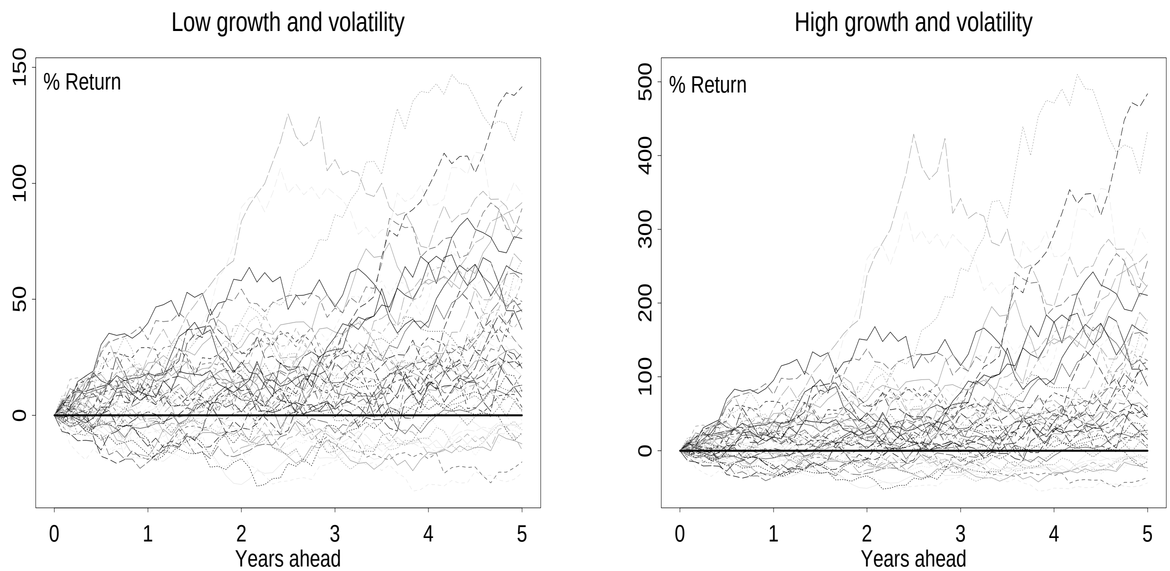

Figure 1.5 *Simulations of accumulated equity return from geometric random walk (monthly time scale).*

scenarios in Figure 1.5 are monthly and apply to the k-step returns $R^*_{0:k} = S^*_k/S_0 - 1$ rather than the share price directly (the intial value $S_0 = s_0$ is then immaterial). Values for the parameters were

$\xi = 0.4\%$, $\sigma = 4\%$ — *low yield and risk*

$\xi = 0.8\%$, $\sigma = 8\%$, — *high yield and risk*

both possible in real life. The potential for huge gain *and* huge loss is enormous. After five years up to 50% of the orginal capital is lost in some of the scenarios on the right! Be aware that scales on the vertical axes differ so that the uncertainty of the first model scenario is around one third of the other. The performance of equity is wild and unstable and very different from money market assets.

1.6 Bibliographical notes

General references For ideas on the practical side of the actuarial profession, try Szabo (2004). A general introduction to assets and liabilities is Booth, Chadburn, Cooper, Haberman and James (1999) with management issues covered in Williams, Smith and Young (1998). Neither of those make much use of mathematics, but it is more of that in Panjer (1998) which is a collection of articles by different authors. The encyclopedia edited by Teugels and Sundt (2004) is a broad review of actuarial science and its history with more weight on mathematical than computational ideas. Entirely devoted to history, the ancient one included, is the ten volumes edited by Haberman and Sibbet (1995) and the landmark papers on pension insurance in Bodie and Davis (2000). Useful, elementary reviews of traditional life and property insurance mathematics are Gerber (1997), Promislow (2006) and Boland (2007). There are countless introductions to financial risk that take the material in Sections 1.3 and 1.4 much further. Mathematics is kept on a fairly elementary level in Copeland, Weston and Shastri (2005) and Hull (2006), the latest edtitions of two classics. An exceptionally clear outline of investment risk is Luenberger (1998), see also Danthine and Donaldson (2005).

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Monte Carlo Simulation was established as a central method of science during the last decades of the twentieth century. The term means two different things and is in applied physisics and certain parts of engineering often associated with numerical solutions of partial differential equations; see Langtangen (2003). Apart from powerful computers being needed this has little to do with the stochastic version employed here. Stochastic simulation (or Monte Carlo) is an enormously versatile technique. It was used to reconstruct global processes in Casti (1998) and networks of economic agents in Levy, Levy and Solomon (2000); see Bølviken (2004) for other references. Applications in the present book are less exotic, although there will be many extensions of the proceesses simulated in Section 1.5. Algorithms will be presented as in Algorithm 1.1 which is written in a pseudo-code taken from Devroye (1986). Although the Monte Carlo metod is well established in actuarial science, few textbooks apart from a Daykin, Pentikäinen and Pesonen (1994) integrate the technique deeply. This is different in financial economy where a number of such books have been written. Elementary introductions are Vose (2000) and Chan and Wong (2006) with Lyuu (2002) being more advanced. In statistics Gentle (2002). Robert and Casella (2004) and Ripley (2006) are reviews of Monte Carlo-based methodology with Gentle, Härdle and Mori (2004) being a collection of articles over a broad spectrum of themes. Ross (1997) and Fishman (2001) and (2006) are introductions to stochastic simulation in operations research.

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