sol

$$Y_{ij} = \underbrace{\mu_i}_{} + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2) \qquad ①$$

group means

$$i, j \in I, J \quad \|I\| = 3, \quad \|J\| = 3$$

$$I = \{1, 2, 3\}, \quad J = \{1, 2, 3\}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_9 \end{pmatrix} = X\beta + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} + \vec{\varepsilon}$$

$$\varepsilon^T \varepsilon \xrightarrow[\beta]{} \min$$

$$(Y - X\beta)^T (Y - X\beta) \xrightarrow[\beta]{} \min$$

$$\left((Y - X\beta)^T(-2X)\right) = 0 \iff \text{②}$$

$$X^T(Y - X\beta) = 0.$$

$$X^T Y = X^T X \beta \iff \boxed{\hat\beta_{opt} = (X^T X)^{-1} X^T Y}$$

$$\hat\beta_{opt} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

$$Y_{ij} = \mu + a_i + \varepsilon_{ij} \quad , \varepsilon_{ij} \sim N(0,\sigma^2)$$

μ - global mean

a_i - deviations

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$$\sum_{i \in I} a_i = 0$$ - constraints

$$\underline{Y} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & \vdots \\ 1 & \vdots & 1 & \vdots \\ \vdots & & \vdots & 0 \\ \vdots & & 0 & 1 \\ 1 & 0 & 0 & 1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \mu \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} + \varepsilon = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

(*)

s.t. $\sum a_i = 0$

$a_1 + a_2 + a_3 = -3\mu + \sum \mu_i = 0$

$\bar{\mu} = \frac{1}{3} \sum \mu_i$ //

$$= \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ 1 & 0 & \vdots \\ 1 & 1 & \vdots \\ \vdots & 1 & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \vec{\mu} + \vec{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 \\ \vdots & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & 1 \\ 0 & 0 & \vdots \end{pmatrix} \begin{pmatrix} a_1 + \mu \\ a_2 + \mu \\ a_3 + \mu \end{pmatrix} + \vec{\varepsilon}$$

$$\hat{\beta}_{OLS} = \begin{pmatrix} a_1 + \mu \\ a_2 + \mu \\ a_3 + \mu \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

$$\vec{Y} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ a_1 \\ a_2 \end{pmatrix} + \vec{\varepsilon} = W\gamma + \vec{\varepsilon} \quad (1)$$

— on the other hand

$a_3 = -a_1 - a_2$

from constraints

$=: W$

$$\hat{\gamma} = (W^T W)^{-1} W^T Y$$

vs

$$\beta = (X^T X)^{-1} X^T Y$$

$\boxed{X\beta = W\gamma} \Leftrightarrow$

$\exists E :$

$\exists T, S^T$

$X = TW$

$X = WS^T$

$\cancel{W} S^T \beta = \cancel{W}\gamma$

$S^T \beta = \gamma$

$X\hat{\beta} = X(\cancel{X^T X})^{-1} \cancel{X^T} X X^{-1} y =$

$= y \Leftrightarrow$

$\hat{\gamma} = (W^T W)^{-1} W^T Y = (W^T W)^{-1} W^T X \hat{\beta} \Leftrightarrow$

$\Leftrightarrow S^T = (W^T W)^{-1} W^T X$

alternatively

$X\beta = W\gamma$

$\gamma = (W^{-1}) X\beta$

$S^T = (W^{-1}) X$

pseudo-inverse of W

can be less efficient computationally

$$Y_{ij} = \mu + \gamma_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2) \quad (5)$$

mean of group 1 — deviations from μ — $\gamma_1 = 0$ — constraint

$$Y = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & 1 & 0 \\ & 1 & 0 \\ & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}}_{=U} \underbrace{\begin{pmatrix} \mu \\ \gamma_2 \\ \gamma_3 \\ \vdots \end{pmatrix}}_{=Z} + \vec{\varepsilon}$$

$$\hat{Z} = (U^T U)^{-1} U^T Y$$

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$H_0: Ca = 0$

$H_1: Ca \neq 0$

$C \mathbf{x} (\hat{\beta}) = 0 \Rightarrow$ whether $\mu_1 = \mu_2 = \mu_3 = 0$

$C \mathbf{x} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$ $\quad$ $C \mathbf{x} \begin{pmatrix} \beta_1 - \mu \\ \beta_2 - \mu \\ \beta_3 - \mu \end{pmatrix} = 0$

whether $a_1 = a_2 = a_3 = 0$

$\Rightarrow \mu_1 = \mu_2 = \mu_3 = \mu$!

SO2

$$(1)\ Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \quad \varepsilon_i \sim N(0,\sigma^2) \quad (1)$$

$$(2)\ Y_i = \underline{a + b}\,(\overbrace{X_i - \bar{X}}^{?}) + \varepsilon_i, \quad \varepsilon_i \sim N(0,\sigma^2)$$

$$\bar{X} = \frac{1}{n}\sum X_i \,. \qquad \boxed{\beta_0 = a - b\bar{X}, \quad \beta_1 = b}$$

reparametrization

MLE is invariant to it, OLS in this case too!

$$(1) \Longleftrightarrow Y = X\vec{\beta} + \vec{\varepsilon} = \qquad X := \begin{pmatrix} 1 & X_1 \\ 1 & \vdots \\ \vdots & \vdots \\ 1 & \\ 1 & X_n \end{pmatrix}$$

$$= X\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \vec{\varepsilon} = X\begin{pmatrix} a - b\bar{X} = \beta_0 \\ b = \beta_1 \end{pmatrix} + \vec{\varepsilon}$$

$$\hat{\beta}_{OLS} = (X^T X)^{-1} \cdot X^T Y$$

$$(2) \Rightarrow Y = \begin{pmatrix} 1 & X_1^* = X_1 - \bar{X} \\ 1 & X_2^* = X_2 - \bar{X} \\ \vdots & \vdots \\ 1 & X_n^* = X_n - \bar{X} \end{pmatrix} \gamma + \vec{\varepsilon} = \quad (2)$$

$W =$

$$= W\gamma + \vec{\varepsilon}, \quad \gamma = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\hat{\sigma}^2 = \frac{(Y - X\hat{\beta})^T (Y - X\hat{\beta})}{n - p}$$

n: # observ; p: # covariates

$$\hat{\gamma} = (W^T W)^{-1} W^T Y = S^T \hat{\beta}!$$

$$\Sigma(\hat{\beta}, \hat{\beta}) = \sigma^2 (X^T X)^{-1} \approx \hat{\sigma}_{\hat{\beta}}^2 (X^T X)^{-1}$$

$$\Sigma(\hat{\gamma}, \hat{\gamma}) = \sigma^2 (W^T W)^{-1} \approx \hat{\sigma}_{\hat{\gamma}}^2 (W^T W)^{-1}$$

$$\operatorname{cov}(\hat{a}, \hat{b}) = 0 = \Sigma(\hat{r}, \hat{r})_{1,2} =$$ ③

$$\sigma^2 \left(\begin{pmatrix} 1 & 1 & \cdots & 1 \\ X_1 - \bar{X} & \cdots & \cdots & X_n - \bar{X} \end{pmatrix} \begin{pmatrix} 1 & X_1 - \bar{X} \\ 1 & \vdots \\ \vdots & \vdots \\ 1 & X_n - \bar{X} \end{pmatrix} \right)_{1,2} =$$

!

$\bar{X} = \frac{1}{n} \Sigma X_i$

$$= \sigma^2 \sum_i (X_i - \bar{X}) = \sigma^2 \Big(\underset{\substack{\| \\ n\bar{X}}}{\Sigma X_i} - n\bar{X} \Big) = 0$$

~~S_{03}~~ S_3 Two way **ANOVA** (1)

e.g.
I - groups J - experiments.

& N - repeats of experiments from the group.

$\|J\| = 3$, $\|I\| = 3$, $\|N\| = 1$

(1) $Y_{ijk} = \underbrace{\mu}_{\text{general mean}} + \underbrace{a_i}_{\text{group effect}} + \underbrace{\beta_j}_{\text{exper. effect}} + \underbrace{\delta_{ij}}_{\text{mixed effect}} + \underbrace{\varepsilon_{ijk}}_{\text{rand. effect}}$,

constr. $\boxed{\sum a_i = 0\ \sum \beta_j = 0,\ \sum_i \delta_{ij} = 0,\ \sum_j \delta_{ij} = 0}$ $\varepsilon_{ijk} \sim N(0, \sigma^2)$

(2) $Y_{ijk} = \underbrace{\gamma}_{\text{group 1 mean}} + \underbrace{K_i}_{\text{group deviation}} + \underbrace{\rho_j}_{\text{exper. deviation}} + \underbrace{\tau_{ij}}_{\text{mixed deviation from } \gamma} + \underbrace{\varepsilon_{ijk}}_{\text{rand. effect}}$,

$\boxed{K_1 = 0,\ \rho_1 = 0,\ \tau_{1j} = 0\ \forall j \in J,\ \tau_{i1} = 0\ \forall i \in I}$ $\varepsilon_{ijk} \sim N(0, \sigma^2)$

constraints

(1) $\mu + a_1 + \beta_1 + \delta_{11} = \gamma + k_1 + \rho_1 + \tau_{11}$ 2

9 equations, 16 unknowns

⋮

(9) $\mu + a_3 + \beta_3 + \delta_{33} = \gamma + k_3 + \rho_3 + \tau_{33}$

(10) $k_1 = 0$ (12) $\tau_{11} = 0$ (14) $\tau_{31} = 0$

(11) $\rho_1 = 0$ (13) $\tau_{12} = 0$ (15) $\tau_{21} = 0$

(16) $\tau_{13} = 0$

now OK!

$$Y = X \beta + \vec{\varepsilon} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & & & & & & & & & & \\ \vdots & & & & & & & & & & \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & & 1 \end{pmatrix} \begin{pmatrix} \mu \\ a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \\ \delta_{11} \\ \vdots \\ \delta_{33} \end{pmatrix} + \vec{\varepsilon} =$$

singular

(3)

$$\delta_{31} = -\delta_{11} - \delta_{21}$$

for δ_{33} s.t. constraint (1)

$-\delta_{12} - \delta_{11}$

$$\delta_{33} = -\delta_{13} - \delta_{23}$$
$$\delta_{33} = -\delta_{31} - \delta_{32}$$

$$= \begin{array}{c} \\ \\ \\ \\ a_{31} \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} \mu & a_1 & a_2 & b_1 & b_2 & \delta_{11} & \delta_{21} & \delta_{12} & \delta_{22} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ \vdots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \cdot & -1 & -1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \begin{pmatrix} \mu \\ a_1 \\ a_2 \\ b_1 \\ b_2 \\ \delta_{11} \\ \delta_{21} \\ \delta_{12} \\ \delta_{22} \end{pmatrix}$$

||

X

full rank

$X^T X$ - block diagonal ☺

easy to invert!!!

Give in SO1

$$Y = W\gamma^* + \vec{\varepsilon}$$

W^TW - (4)
not sparse
⇓
difficult to invert

$$Y = \begin{array}{c} \\ 11 \\ 21 \\ 31 \\ 12 \\ 13 \\ 22 \\ 32 \\ 23 \\ 33 \end{array}
\left(\begin{array}{ccccccccc}
\gamma & K_2 & K_3 & \rho_2 & \rho_3 & \tau_{22} & \tau_{32} & \tau_{23} & \tau_{33} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
\begin{pmatrix} \gamma \\ K_2 \\ K_3 \\ \rho_2 \\ \rho_3 \\ \tau_{22} \\ \tau_{32} \\ \tau_{23} \\ \tau_{33} \end{pmatrix} + \vec{\varepsilon}$$

easy to construct

but:

$$\boxed{\gamma = S^T\beta}$$ still true

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