19.11.2015 S13 (new set), ex 20

S13 new $\quad y_i \sim f(y|\mu,\nu) = \frac{1}{\Gamma(\nu)} y^{\nu-1} \left(\frac{\nu}{\mu}\right)^{\nu} \exp\left(-\frac{\nu y}{\mu}\right), \mu,\nu,y>0$

$\eta = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$

a) show $2n(\log(\hat{\nu}) - \psi(\hat{\nu})) = \frac{\Delta}{\hat{\nu}}$, where

$\psi(\nu) = \frac{\Gamma'(\nu)}{\Gamma(\nu)}$

estimation of $\vec{\beta}$ and φ are independent in terms of $\vec{\beta}$ can be estimated without knowing φ, (see S5)

$f(y_1,\dots,y_n|\mu_1,\dots,\mu_n,\nu) = [\text{independence}] =$

$= \prod_{i=1}^{n} f(y_i|\mu_i(\vec{\beta},x_i),\nu) = \frac{1}{(\Gamma(\nu))^n}(y_1\times\dots\times y_n)^{\nu-1} \frac{\nu^{n\nu}}{(\mu_1\times\dots\times\mu_n)^{\nu}} \times$

$\times \exp\left(-\nu \sum_{i=1}^{n} \frac{y_i}{\mu_i}\right)$.

$l(\mu_1,\dots,\mu_n,\nu) = \log(f(y_1,\dots,y_n|\mu_1,\dots,\mu_n,\nu)) =$

$= -n\log\Gamma(\nu) + (\nu-1)\sum_{i=1}^{n}\log y_i + n\nu\log\nu - \nu\sum_{i=1}^{n}\log\mu_i -$

$- \nu\sum_{i=1}^{n}\frac{y_i}{\mu_i}$

1) we estimate $\hat{\vec{\beta}}$ and correspondingly $\hat{\vec{\mu}}$

2) estimate variance parameter ν

$\frac{\partial l(\cdot)}{\partial \nu} = -n \cdot \frac{1\,\Gamma'(\nu)}{\Gamma(\nu)} + \sum_{i=1}^{n}\log y_i + n\log\nu + \frac{n\cancel{\nu}}{\cancel{\nu}} -$

$- \sum_{i=1}^{n}\log\mu_i - \sum_{i=1}^{n}\frac{y_i}{\mu_i} = \left[\psi(\nu) = \frac{\Gamma'(\nu)}{\Gamma(\nu)}\right] =$

$= -n\psi(\nu) + \sum_{i=1}^{n}\log y_i + n + n\log\nu - \sum_{i=1}^{n}\log\mu_i - \sum_{i=1}^{n}\frac{y_i}{\mu_i} = 0$

$\Leftrightarrow 2n(\log(\nu) - \psi(\nu)) = 2\left(\sum_{i=1}^{n}\log\mu_i + \sum_{i=1}^{n}\frac{y_i}{\mu_i} - \sum_{i=1}^{n}\log y_i - n\right) =$

$= 2\left(\sum_{i=1}^{n}\left(\log\frac{\mu_i}{y_i} + \frac{y_i}{\mu_i} - \frac{\mu_i}{\mu_i}\right)\right) = \frac{\nu 2}{\nu}\sum_{i=1}^{n}\left(\frac{y_i-\mu_i}{\mu_i} - \log\frac{y_i}{\mu_i}\right) =$

$\left[\Delta = 2\nu\sum_{i=1}^{n}\left(\frac{y_i-\mu_i}{\mu_i} - \log\frac{y_i}{\mu_i}\right)\right] = \frac{\Delta}{\nu} \Leftarrow)$

$2n(\log(\hat{\nu}) - \psi(\hat{\nu})) = \frac{\Delta}{\hat{\nu}}$

b) Show that assymptotic distributions of $\hat{\vec{\beta}}$ & $\hat{\nu}$ are independent.

$(\vec{\beta},\nu) := \theta, \quad \hat{\theta}_{MLE} \xrightarrow[n\to\infty]{d} N_{p+2}(\theta, \Sigma_\theta) =$

$= N_{p+2}(\theta, I_\theta^{-1}) = N_{p+2}\left(\begin{pmatrix}\beta_0\\ \vdots\\ \beta_p\\ \nu\end{pmatrix}, \begin{pmatrix} \mathrm{Var}\,\hat{\beta}_0 & \mathrm{cov}(\hat{\beta}_0,\hat{\beta}_1) \dots & \mathrm{cov}(\hat{\beta}_0,\hat{\nu}) \\ & \ddots & \\ & \mathrm{Var}\,\hat{\beta}_p & \\ & & \mathrm{Var}\,\hat{\nu}\end{pmatrix}\right)$

recall S6 in general form we show that

$\frac{\partial l(\cdot)}{\partial \beta_j} = \frac{1}{\varphi}\sum_{i=1}^{n}\left(\frac{y_i-\mu_i}{1}\right)\frac{x_{ij}}{\dot{g}(\mu_i)V(\mu_i)} \Rightarrow$

$J_{\beta_j,\varphi} = \frac{\partial^2 l(\cdot)}{\partial\beta_j\partial\varphi} = -\frac{1}{\varphi^2}\sum_{i=1}^{n}\left(\frac{y_i-\mu_i}{1}\right)\frac{x_{ij}}{\dot{g}(\mu_i)V(\mu_i)}$

$I_{\beta_j,\varphi} = E\left\{-\frac{\partial^2 l(\cdot)}{\partial\beta_j\partial\varphi}\right\} = E\left(\frac{1}{\varphi^2}\sum_{i=1}^{n}\frac{(y_i)-\mu_i}{1}\frac{x_{ij}}{\dot{g}(\mu_i)V(\mu_i)}\right)$

$= \frac{1}{\varphi^2}\sum_{i=1}^{n}\frac{E(y_i)-\mu_i}{1}\frac{x_{ij}}{\dot{g}(\mu_i)V(\mu_i)} = [E(y_i) = \mu_i] =$

$= 0$

$I_\theta = \begin{pmatrix} I_{\hat{\beta}} & 0 \\ 0 & I_{\hat{\varphi},\hat{\varphi}}\end{pmatrix}$

$\Sigma_\theta = I_\theta^{-1} = \begin{pmatrix} I_{\hat{\beta}} & 0 \\ 0 & I_{\hat{\varphi}\hat{\varphi}}\end{pmatrix}^{-1} = \begin{pmatrix} I_{\hat{\beta}}^{-1} & 0 \\ 0 & I_{\hat{\varphi},\hat{\varphi}}^{-1}\end{pmatrix}$

$\Rightarrow \mathrm{cov}(\hat{\beta}_j,\hat{\varphi}) = 0 \Rightarrow$ [θ is asymptotically MVN] $\Rightarrow f(\hat{\beta}_j,\hat{\varphi}) = f(\hat{\beta}_j)f(\hat{\varphi})$

$\forall j = 0,\dots,p$ (independence)

ex 20

$z = \log y \sim N(\mu, \sigma^2)$

$f_z(z) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}(z-\mu)^2\right)$.

find the density of y.

$y = \exp(z)$

$f_Y(y) = f_z(z(y)) \left| \frac{\partial z(y)}{\partial y} \right| =$

$= [z(y) = \log(y)] = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}(\log(y)-\mu)^2\right) \times$

$\times \frac{1}{y} = f_{(z)}(z(y)) \cdot \frac{1}{y}$ - density of Y

$= f_Y(y) = f_z(z)/y$ express new one through original

$\log L(\mu, \sigma^2 | z) = \log \prod_{i=1}^{n} f_z(z_i) =$

$= \sum_{i=1}^{n} \log f_z(z_i) = \sum_{i=1}^{n} \log(f_Y(y_i) \cdot y_i) =$

$= \sum_{i=1}^{n} [\log f_Y(y_i) + \log y_i] = \underbrace{\sum_{i=1}^{n} \log f_Y(y_i)}_{\text{log-likelihood for } y_i} + \sum_{i=1}^{n} \log y_i$

$\Leftrightarrow \ l(\mu, \sigma^2 | y_1, \dots, y_n) = \underbrace{l(\mu, \sigma^2 | z_1, \dots, z_n)}_{\text{we know from LM.}} - \sum_{i=1}^{n} \log y_i =$

$= l(\mu, \sigma^2 | z_1, \dots, z_n) - \sum_{i=1}^{n} z_i$

$\mathrm{Var}(y) = \varphi \ddot{a}(\theta) = \varphi \dot{\mu}(y) = \frac{1}{V} \dot{\mu}(y) =$

j) $\hat{\varphi} = \frac{X^2}{n-q}$, $X^2 = \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{V(\hat{\mu}_i)} = \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{\hat{\mu}_i^2} =$

$V(\hat{\mu}_i) = \ddot{a}(\theta_i)$.

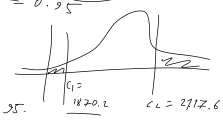
$= \varphi \cdot X^2_{n-q}$

$\mathrm{var}(y_i) = \ddot{a}(\theta_i)\varphi = V(\hat{\mu}_i) \cdot \varphi \Leftrightarrow$

$\hat{\varphi} = \frac{\widehat{\mathrm{var}}(y_i)}{V(\hat{\mu}_i)}$

$P\left(c_1 < \frac{X^2}{\varphi} < c_2\right) = 1 - \alpha = 0.95$

for X^2_{1992}, we have



$P\left(\frac{X^2}{c_1} > \varphi > \frac{X^2}{c_2}\right) = 0.95$.

$\frac{X^2}{c_1} = 0.0279$ $\qquad$ $\frac{X^2}{c_2} = 0.02463$

$\varphi \in [0.02463, 0.02790]$

j) if φ was MLE estimated.

$\begin{pmatrix} \hat{\vec{\beta}} \\ \hat{\varphi} \end{pmatrix} \sim N_{p+2}\left(\begin{pmatrix} \vec{\beta} \\ \varphi \end{pmatrix}, \Sigma_{\vec{\beta}, \varphi} \right) \Rightarrow$

$f(y | \hat{\vec{\beta}}, x_{new}, \hat{\varphi})$ - transformation of $\hat{\vec{\beta}}, \hat{\varphi}$, or the transformation of $\hat{\theta}_{MLE}$

$\boxed{\hat{f}(y | \hat{\vec{\beta}}, x_{new}, \hat{\varphi})} \xrightarrow[n \to \infty]{} N_1(\mu, \sigma^2)$

comment:

note that this $\hat{\varphi}$ is **not MLE** $\Rightarrow l(\hat{\beta}^*_{MLE}, \hat{\varphi}^*_{MLE}) \geq l(\hat{\beta}_{MLE}, \hat{\varphi})$ and

$l(\tilde{\vec{\beta}}^*_{MLE}, \tilde{\varphi}^*_{MLE}) \geq l(\tilde{\beta}_{MLE}, \tilde{\varphi})$ and

$\Delta_1 = 2(l(\tilde{\vec{\beta}}^*_{MLE}, \tilde{\varphi}^*_{MLE}) - l(\hat{\vec{\beta}}^*_{MLE}, \hat{\varphi}_{MLE}))$ can

be different from

$\Delta_2 = 2(l(\tilde{\vec{\beta}}_{MLE}, \tilde{\varphi}) - l(\hat{\beta}_{MLE}, \hat{\varphi}))$!!!

$\Rightarrow$ one has to be accurate with deviance in this problem! $\quad \Delta_1 ?? \Delta_2$