10.09.2015 / ex 8 - 11 (ex 8) (1)

$f_0(y)$, $M(t) = \exp(a(t)) < +\infty$

$f_Y(y|\theta) \propto \exp(\theta y) f_0(y)$

a) $f_Y(y|\theta) = K \exp(\theta y) f_0(y)$

$\int_{\Omega_y} f_Y(y|\theta)\, dy = 1 \iff \int_{\Omega_y} K \exp(\theta y) f_0(y)\, dy = 1$

$\iff K \underbrace{\int_{\Omega_y} \exp(\theta y) f_0(y)\, dy}_{M(t)} = K M(t) =$

$= K \exp(a(t)) = 1 \iff \boxed{M(t) < +\infty} \iff K = \exp(-a(t))$

sep 10-10:02

b) $f_0(y) \propto \frac{1}{y!}$ find $f_Y(y|\theta)$ (2)

$f_Y(y|\theta) \propto \exp(\theta y) \frac{1}{y!} \iff f_Y(y|\theta) = K \exp(\theta y) \frac{1}{y!}$

$\sum_{y=0}^{+\infty} \left(K \frac{e^{\theta y}}{y!}\right) = 1$ $\quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \iff$

$K e^{e^\theta} = 1 \iff \left[\theta = \log \mu\right] \iff$

$\iff K = \exp(-\exp(\theta)) = e^{-\mu}$

$f_Y(y|\theta) = \frac{e^{-\mu}}{y!} e^{y \log \mu} = \frac{e^{-\mu} \left(e^{\log \mu}\right)^y}{y!} =$

$\frac{e^{-\mu} \mu^y}{y!}$ — poisson distribution

sep 10-10:08

ex 9) $Y_1, \dots, Y_n \sim$ i.i.d. $f_Y(y|\theta, \varphi)$ from expo. family

3

$$M_{Y_i}(t) = e^{\frac{a(\theta + t\varphi) - a(\theta)}{\varphi}}$$

$$M_{\Sigma Y_i}(t) = E\left\{ e^{\Sigma y_i t} \right\} = E\left\{ \prod e^{y_i t} \right\} = \left[y_i \sim \text{i.i.d} \right] = \prod E\left\{ e^{y_i t} \right\} = [\text{iid}] =$$

$$= \left(E\left\{ e^{yt} \right\} \right)^n = \left(e^{\frac{a(\theta + t\varphi) - a(\theta)}{\varphi}} \right)^n =$$

$$= e^{\frac{n a(\theta + t\varphi) - n a(\theta)}{\varphi}} = \left[a^*(x) = n a(x) \right] =$$

$$= e^{\frac{a^*(\theta + t\varphi) - a^*(\theta)}{\varphi}} \;//$$

MGF for exponential family too!

sep 10-10:29

$S = \Sigma Y_i$

$\bar{Y} = \frac{S}{n}$ $\boxed{s(y) = ny}$

$$g_{\bar{Y}}(y) = \left[\text{transformation of rV formula} \right] = f_S(s(y)) \left| \frac{\partial s(y)}{\partial y} \right| =$$

4

$$= C(s(y), \varphi)\, e^{\frac{s(y)\theta - a^*(\theta)}{\varphi}} \left| \frac{\partial ny}{\partial y} \right| =$$

$$= C(s(y), \varphi)\, e^{\frac{ny\theta - n a(\theta)}{\varphi}} \cdot n =$$

$$= C(s(y), \varphi)\, e^{\frac{y\theta - a(\theta)}{\frac{\varphi}{n}}} \cdot n =$$

$$\left[\varphi^* = \frac{\varphi}{n} \right] = C\left(s(y), \frac{\varphi}{n}\right) e^{\frac{y\theta - a(\theta)}{\frac{\varphi}{n}}} =$$

$$= C^*(y, \varphi^*)\, e^{\frac{y\theta - a(\theta)}{\varphi^*}}$$ - exponent. family

X - r.v. $\sim p(x)$ $\quad y = f(x)$ - r.v. $\sim g(y)$ $\quad x = f^{-1}(y) = x(y)$

$$\boxed{g(y) = p(x(y)) \left| \frac{\partial x(y)}{\partial y} \right|}$$

r.v. transformation

sep 10-10:36

8) $Y_1, \dots, Y_n \sim Bin(1, \mu)$ find $\bar{Y}$ distribution (5)

$$f(y|\mu) = \frac{1}{y!(1-y)!} \mu^y (1-\mu)^{1-y} = \left[\begin{array}{l} y! = 1 \\ (1-y)! = 1, \\ \text{since } y \in \{0,1\} \end{array} \right] = \mu^y (1-\mu)^{1-y}$$

$$M_Y(t) = (1 - \mu + \mu e^t)^2$$

$$M_{\Sigma Y_i}(t) = (M_Y(t))^n = (1 - \mu + \mu e^t)^n -$$

Mgf for Binomial r.v. with Binomial(n, μ)

$$S = \Sigma y_i \sim Bin(n, \mu)$$

$$f_S(s|n,\mu) = \binom{n}{s} \mu^s (1-\mu)^{n-s}$$

$$g(s) = \frac{s}{n} = \bar{y} - \text{new r.v.}$$

sep 10-10:46

(6)

$$f_{\bar{Y}}(\bar{y}|n,\mu) = f_S(s(\bar{y})|n,\mu) \left| \frac{\partial s(\bar{y})}{\partial \bar{y}} \right| =$$

$$s(\bar{y}) = n\bar{y}$$

$$= \binom{n}{n\bar{y}} \mu^{n\bar{y}} (1-\mu)^{n-n\bar{y}} \cdot n$$

what is: $\lim\limits_{n\to\infty} f_{\bar{Y}}(\bar{y}|n,\mu) = ?$

hint: try applying Stirling formula for $k!$ before finding the limit. Does the result agree with CLT?

additional exercise

sep 10-10:51

ex 10 the rest is on the web! 7

$\underline{I} = cov(S, S) = \Sigma_S$

$a^T \underline{I} a \geq 0 \quad \forall$ non zero a.

I - symmetric

$\underline{I} = V \Lambda V^T \iff a^T \underline{I} a = a^T V \Lambda V^T a \geq 0$

$\Lambda = diag(\lambda_i), \lambda_i \geq 0$

$\lambda_i \geq 0$

V_i are orthogonal.

$V^T V = V V^T = I_n = diag(1)$

$\Downarrow$

$$\det(\Sigma_S) = \det(\underline{I}) \geq 0$$

$$a^T \underline{I} a = a^T cov(S, S) a^T = cov(a^T S, a^T S) =$$

$$= \left[\begin{array}{l}\text{in this case} \\ \text{since } E(S) = \vec{0}\end{array}\right] = E(a^T S (a^T S)^T) =$$

$$E(a^T S S^T a) = a^T E(S S^T) a \geq 0$$

sep 10-10:54

ex11 $Y = (Y_1, \dots, Y_p)^T \sim N_p(\vec{\mu_Y}, \Sigma_Y)$ 8

$Y = AZ + \mu$, where $Z = (Z_1, \dots, Z_p)^T$

Z_i are i.i.d. $N(0, 1)$

$\mu = (\mu_1, \dots, \mu_p)^T \qquad A_{p \times p}$

$$\mu_Y \mid E(Y) = A E(Z) + E(\mu) =$$

$$= \vec{0} + \mu$$

$$\Sigma_Y = cov(Y, Y) = cov(AZ, AZ) +$$

$$+ \underbrace{cov(\mu, \mu)}_{\text{constant}} = A \Sigma_Z A^T + 0 = A A^T \iff Y \sim N_p(\mu, AA^T)$$

$$\Sigma_Z = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

sep 10-11:03

b) B - non singular matrix $p \times p$

$V = BY$ - also MVN

$V = BY = B(Az + \mu) =$

$= (BA)z + B\mu \sim N_p(B\mu, (BA)(BA)^T) =$

$= N_p(B\mu, BAA^T B^T) =$

$= N_p(B\mu, B\Sigma_Y B^T)$

9

sep 10-11:23

c) $Y_1 = Az + \mu$ & $Y_2 = Lz + \mu$ (10)

$AA^T = LL^T$, have the same distribution

L - lower triangular

$\Sigma_{Y_1} = AA^T \Rightarrow \exists L : \Sigma_{Y_1} = LL^T$

$E(Y_2) = L\underbrace{E(z)}_{0} + \mu = \mu$

$\Sigma_{Y_2} = L\underbrace{cov(z,z)}_{diag(1)}L^T + 0 =$

$= LL^T = \Sigma_{Y_1} = AA^T$

$Y_1, Y_2 \sim N_p(\mu, AA^T) = N_p(\mu, LL^T)$

sep 10-11:26

(11)

d) $B_{q \times p}$; $q \le p$ $B_{ij} = 1$, $j = i$, 0 otherwise

$V = BY$ - MVN

$$V = BY = B(LZ + \mu) = BL_{p\times p}Z + B\mu =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & \dots & 1 & \dots & 0 \end{pmatrix}\begin{pmatrix} L_{11} & & & \\ L_{21} & L_{22} & & 0 \\ \vdots & & \ddots & \\ L_{p1} & \dots & & L_{pp} \end{pmatrix} Z + B\mu =$$

and etc.

$$= \begin{pmatrix} L_{11} & 0 & \dots & 0 \\ L_{21} & L_{22} & 0 \dots & 0 \\ \dots \\ L_{q1} & \dots & & L_{qq} \end{pmatrix}\begin{pmatrix} Z_1 \\ \vdots \\ Z_q \end{pmatrix} + \mu_{1:q} =$$

actually $BLZ =$

$$\begin{pmatrix} L_{11} & \dots & 0 & 0 \\ \vdots & & & \\ L_{q1} & \dots & L_{qq} & 0 \end{pmatrix}\begin{pmatrix} Z_1 \\ \vdots \\ Z_p \end{pmatrix}$$

all other q are 0

sep 10-11:30

(12)

$$= L^*_{q\times q} Z^*_{q\times 1} + \mu^*_{q\times 1} \sim N_q(\mu^*, L^* L^{*T})$$

, by c)

e) $B_{q\times p}$ with rank $q \le p$.

$V = BY \sim$ MVN

$\tilde{B} = \begin{pmatrix} B \\ B^{(2)} \end{pmatrix}$ $\tilde{B}_{p\times p}$ is now $p \times p$

$\tilde{B}$ - non singular

$$\tilde{V} = \tilde{B} Y \sim N_p(\tilde{B}\mu, \tilde{B} A A^T \tilde{B}^T) =$$

by b)

sep 10-11:36

13

$$\tilde{V} = \begin{pmatrix} V_{1:q} \\ V^{(2)}_{q+1:p} \end{pmatrix} \sim N_p\left(\begin{pmatrix} B\mu \\ B^{(2)}\mu \end{pmatrix}, \Sigma_{\tilde{V}} \right)$$

$$V_{1:q} = \begin{array}{c|c} & \begin{array}{ccccccccc} 1 & & & & q & & & & p \end{array} \\ \hline \begin{array}{c} 1 \\ \vdots \\ q \end{array} & \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 & 0 & \cdots & 0 \\ - & - & - & - & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \tilde{V} \end{array}$$

as we have shown in d)

$V_{1:q}$ is Multivariately Normal

10.09.2015