24.09.2015 | ex 5.8 - 5.11 ; ex 5.3 , 5.5 | ①

ex 5.9 a) Y_1, Y_2, Y_3 are independ. Poisson with means $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, $\lambda_3 = \mu_3$

$P(Y_1 = z \mid Y_1 + Y_2 = y)$?

$$M_{Y_1+Y_2}(t) = E\{e^{t(y_1+y_2)}\} = E\{e^{ty_1} e^{ty_2}\} =$$

$$= [y_1 \,\&\, y_2 \text{ are ind}] = E\{e^{ty_1}\} E\{e^{ty_2}\} =$$

$$= M_{Y_1}(t) M_{Y_2}(t) = \left[\begin{matrix} M_Y(t) = e^{\lambda(e^t-1)} \\ Y \sim \text{Poiss}(\lambda) \end{matrix}\right] = e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} =$$

$$= e^{(\lambda_1+\lambda_2)(e^t-1)} \Leftrightarrow Y_1 + Y_2 \overset{d}{=} \text{Poiss}(\lambda_1+\lambda_2)$$

$$P(AB) = P(A|B)\,P(B) \Leftrightarrow P(A|B) = \frac{P(AB)}{P(B)}$$

proceed with finding $f(Y_1 = y_1, Y_1 + Y_2 = y)$

we know $f(Y_1 = y_1, Y_2 = y_2) \overset{ind}{=} f(Y_1 = y_1) f(Y_2 = y_2) =$

$$= \frac{\lambda_1^{y_1} e^{-\lambda_1}}{y_1!} \cdot \frac{\lambda_2^{y_2} e^{-\lambda_2}}{y_2!} \text{ ; consider a transformation}$$

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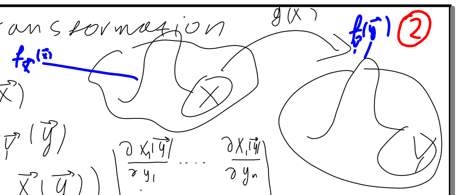
②

$\begin{cases} Z_1 = Y_1 \\ Z_2 = Y_1 + Y_2 \end{cases}$ - transformation



$\vec{X}$; $\vec{Y} = g(\vec{X})$

$\vec{X} \sim f_{\vec{X}}(\vec{x})$ $\quad f_{\vec{Y}}(\vec{y})$

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{x}(\vec{y})) \left| \begin{matrix} \frac{\partial x_1(\vec{y})}{\partial y_1} & \cdots & \frac{\partial x_1(\vec{y})}{\partial y_n} \\ \frac{\partial x_n(\vec{y})}{\partial y_1} & \cdots & \frac{\partial x_n(\vec{y})}{\partial y_n} \end{matrix} \right|$$

$\vec{x}(\vec{y}) = \{x_1(\vec{y}), \ldots, x_n(\vec{y})\}$

$$\begin{matrix} z_1 = y_1 \\ z_2 = y_1 + y_2 \end{matrix} \Leftrightarrow \begin{matrix} y_1(\vec{z}) = z_1 \\ y_2(\vec{z}) = z_2 - z_1 \end{matrix}$$

$$f_{\vec{Z}}(z_1, z_2) = \underbrace{f_{\vec{Y}}(y_1(\vec{z}), y_2(\vec{z}))}_{\text{we have found it}} \left| \begin{matrix} \frac{\partial y_1(\vec{z})}{\partial z_1} & \frac{\partial y_1(\vec{z})}{\partial z_2} \\ \frac{\partial y_2(\vec{z})}{\partial z_1} & \frac{\partial y_2(\vec{z})}{\partial z_2} \end{matrix} \right| =$$

$$= \frac{\lambda_1^{z_1} e^{-\lambda_1}}{z_1!} \frac{\lambda_2^{z_2-z_1} e^{-\lambda_2}}{(z_2-z_1)!} \underbrace{\left| \begin{matrix} 1 & 0 \\ -1 & 1 \end{matrix} \right|}_{= 1\times 1 = 1} =$$

$$= \frac{\lambda_1^{z_1} \lambda_2^{z_2-z_1} e^{-(\lambda_1+\lambda_2)}}{z_1!\,(z_2-z_1)!} \cdot 1 \Rightarrow$$

$$f_{Z_1|Z_2}(z_1|z_2) \overset{\text{Bayes f-la}}{=} \frac{f_{\vec{Z}}(z_1, z_2)}{f_{Z_2}(z_2)} = \left[f_{Z_2}(z_2) \overset{d}{=} \text{Poiss}(\lambda_1+\lambda_2) \right]$$

$$\frac{\lambda_1^{z_1} \lambda_2^{z_2-z_1} \cancel{e^{-(\lambda_1+\lambda_2)}}\, z_2!}{z_1!\,(z_2-z_1)!\, \cancel{e^{-(\lambda_1+\lambda_2)}} (\lambda_1+\lambda_2)^{z_2+z_1-z_1}}$$

- Binomial c.t.

$$= \binom{z_2}{z_1} \underbrace{\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{z_1}}_{p = \frac{\lambda_1}{\lambda_1+\lambda_2}} \underbrace{\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{z_2-z_1}}_{q = 1-p} \overset{d}{=} \text{Binom}\left(n = z_2, p = \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

n

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③

b) $Y_1, Y_2, Y_3 \mid Y_1 + Y_2 + Y_3$ without loss of generality

$Y_1, Y_2, \ldots, Y_n \mid \sum_{i=1}^{n} Y_i$

$$f_{Y_1 \ldots Y_n \mid \Sigma Y_i}(y_1, \ldots y_n \mid \Sigma y_i) = f_{Y_1 \ldots Y_{n-1}}(y_1, \ldots, y_{n-1} \mid \Sigma y_i)$$

we can either proceed with transformation formula for n-dimensional vectors:

$z_1 = Y_1$

$\vdots$

$z_{n-1} = Y_{n-1}$

$z_n = \sum_{i=1}^{n} Y_i$

, since $f_{z_1, \ldots, z_{n-1} \mid z_n}(z_1, \ldots z_{n-1} \mid z_n) = \dfrac{f_{z_1, \ldots z_n}(z_1, \ldots z_n)}{f_{z_n}(z_n)}$

Or just notice that:

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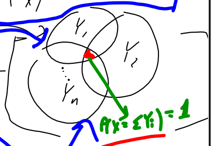
④

$X = \sum_{i=1}^{n} Y_i \overset{d}{=} \text{Poiss}(\sum_{i=1}^{n} \lambda)$

$$f_{Y_1 \ldots Y_n \mid X}(y_1 \ldots, y_n \mid x) = [\text{ Bayes formula }] = \frac{f_{Y_1 \ldots Y_{n-1}, X}(y_1 \ldots, y_{n-1}, x)}{f_X(x)} = \frac{f_{Y_1 \ldots Y_n, X}(y_1 \ldots y_n, x)}{f_X(x)} = \mathbb{1}(x) = \ldots$$



$$= \frac{\lambda_1^{y_1} \ldots \lambda_n^{y_n} e^{-\sum_{i=1}^{n} \lambda_i}}{y_1! \ldots y_n! \, f_X(x)} \cdot P(X = x)$$

$$= \frac{\lambda_1^{y_1} \ldots \lambda_n^{y_n} \cancel{e^{-\Sigma \lambda_i}} (y_1 + \cdots + y_n)!}{y_1! \ldots, y_n! \, (\lambda_1 + \cdots \lambda_n)^{y_1 + \cdots + y_n} \cancel{e^{-\Sigma \lambda_i}}} =$$

$$\frac{(y_1 + \cdots + y_n)!}{y_1! \ldots y_n!} \left(\frac{\lambda_1}{\lambda_1 + \ldots + \lambda_n}\right)^{y_1} \left(\frac{\lambda_2}{\lambda_1 + \cdots + \lambda_n}\right)^{y_2} \cdots \left(\frac{\lambda_n}{\lambda_1 + \cdots \lambda_n}\right)^{y_n}$$

to familiarize

$$= \left[y_1 + \ldots y_n = N, \; \frac{\lambda_i}{\lambda_1 + \ldots \lambda_n} =: p_i \right] = \boxed{\frac{N!}{y_1! \ldots y_n!} \prod_{i=1}^{n} p_i^{y_i}} \overset{d}{=}$$

$\overset{d}{=}$ Multinomial $\left(N = y_1 + \ldots, y_n, \; p_1 = \frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i}, \ldots, p_n = \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i}\right)$

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(S10) $E\{Y\} < +\infty \quad \exists\, f_{X,Y}(x,y) = \underline{p(x,y)}$ (5)

a) $E\{Y\} = E\{E\{Y|X\}\}$

$$E\{Y\} = \int_{\Omega_Y} y\, p(y)\, dy = \int_{\Omega_Y} y \int_{\Omega_X} p(x,y)\, dx\, dy =$$

$$= \left[p(x,y) = p(y|x)\, p(x) \right] = \int_{\Omega_Y} y \int_{\Omega_X} p(y|x)\, p(x)\, dx\, dy =$$

$$= \int_{\Omega_Y} \int_{\Omega_X} y\, p(y|x)\, p(x)\, dx\, dy \overset{\text{change order of integration}}{=} \int_{\Omega_X} \int_{\Omega_Y} y\,(p(y|x)\, p(x)) dy\, dx =$$

$E\{y|x\}$

$$= \int_{\Omega_X} \underbrace{\left[\int_{\Omega_Y} y\, p(y|x)\, dy \right] p(x)\, dx}_{E\{E\{Y|X\}\}} = E\{E\{Y|X\}\}$$

b) $Var\{Y\} = Var\{E\{Y|X\}\} + E\{Var\{Y|X\}\}$ (6)

$$Var\{Y\} = E\{Y^2\} - E^2\{Y\} =$$

$$= \left[\text{conditional expectation formula} \right] = E\{E\{Y^2|X\}\}$$

$$- \left(E\{E\{Y|X\}\} \right)^2 = \left[Var\{Y|X\} = E\{Y^2|X\} - E^2\{Y|X\} \right] =$$

$$E\{Var(Y|X) + E^2\{Y|X\}\} - \left(E\{E\{Y|X\}\} \right)^2$$

$$= E\{Var(Y|X)\} + E\{\underset{z^2}{E^2\{Y|X\}}\} - \left(E\{\underset{z}{E\{Y|X\}}\} \right)^2 =$$

$$E\{z^2\} - E^2(z) = Var(z)$$

$$= E\{Var\{Y|X\}\} + Var\{E\{Y|X\}\}$$

S11 $y \sim NB \iff y|\lambda \sim Poiss(\lambda)$, $\lambda \sim \Gamma amma(\nu, \mu)$ 7 $\Rightarrow$

$$E\{Y\} = E\{E\{Y|\lambda\}\} = E\{\lambda\} = \mu$$

$$Var\{Y\} = E\{Var(Y|\lambda)\} + Var\{E\{Y|\lambda\}\} =$$

$$= E\{\lambda\} + Var\{\lambda\} = \begin{bmatrix} E(\lambda) = \mu \\ var(\lambda) = \frac{\mu^2}{\nu} \end{bmatrix} =$$

$$\mu + \frac{\mu^2}{\nu} = \mu\left(1 + \frac{\mu}{\nu}\right)$$

the second case exactly the same

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ex 5.3 b) 8

$$f(y_i|\nu, \mu_i) \overset{d}{=} Gamma(\nu, \mu_i) =$$

$$= \frac{y_i^{\nu-1}}{\Gamma(\nu)} \left(\frac{\nu}{\mu_i}\right)^{\nu} \exp\left(-\frac{\nu y_i}{\mu_i}\right)$$

$\hat{\mu}_i = g(x_i^T \hat{\beta}) \Rightarrow D := 2 \sum_{i=1}^{n} \left[\log(f(y_i|\nu, \tilde{\mu}_i)) - \log(f(y_i|\nu, \hat{\mu}_i))\right] =$

$\tilde{\mu}_i = y_i$

$$= 2 \sum_{i=1}^{n} (\nu - \cancel{1}) \log y_i - \log \cancel{\Gamma}(\nu) + \nu \cancel{\log} \nu -$$

$$- \nu \log(\tilde{\mu}_i) - \frac{\nu y_i}{\tilde{\mu}_i} - (\nu - \cancel{1}) \log y_i + \log \cancel{\Gamma}(\nu)$$

$$- \nu \cancel{\log} \nu + \nu \log(\hat{\mu}_i) + \frac{\nu y_i}{\hat{\mu}_i} = [\tilde{\mu}_i = y_i] =$$

$$= 2\nu \sum_{i=1}^{n} \log \frac{\hat{\mu}_i}{\underbrace{\tilde{\mu}_i}_{y_i}} - 1 + \frac{y_i}{\hat{\mu}_i} =$$

$$= 2\nu \sum_{i=1}^{n} \log \frac{\hat{\mu}_i}{y_i} + \frac{y_i}{\hat{\mu}_i} - 1$$

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(5.4) $$AR_{Poiss} = \frac{3\left(y_i^{2/3} - \hat{y}_i^{2/3}\right)}{2\,\hat{y}_i^{1/6}} \overset{d}{\approx} \text{Normal}$$ (9)

$$AR := \frac{h(y_i) - h(\hat{y}_i)}{\dot{h}(\hat{y}_i)\sqrt{V(\hat{y}_i)}} = \left[\dot{h}(y) = [V(y)]^{-\frac{1}{3}}\right] = \frac{h(y_i) - h(\hat{y}_i)}{V(\hat{y}_i)^{\frac{1}{6}}}$$

$$f(y_i \mid \lambda_i) = \frac{e^{-\lambda_i}\lambda_i^{y_i}}{y_i!} \Rightarrow V(\mu_i) = V(\lambda_i) = \mu_i$$

$$\dot{h}(y) = y^{-\frac{1}{3}} \Leftrightarrow h(y) = \frac{3}{2}y^{\frac{2}{3}} + k$$

$$AR_{Poiss} = \frac{\frac{3}{2}y_i^{\frac{2}{3}} + \cancel{k} - \frac{3}{2}\hat{y}_i^{\frac{2}{3}} - \cancel{k}}{(\hat{y}_i)^{\frac{1}{6}}} = \frac{\frac{3}{2}\left(y_i^{\frac{2}{3}} - \hat{y}_i^{\frac{2}{3}}\right)}{(\hat{y}_i)^{\frac{1}{6}}} \overset{d}{\approx} \text{Normal} \Rightarrow$$

$\Rightarrow$ easy inference!

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