

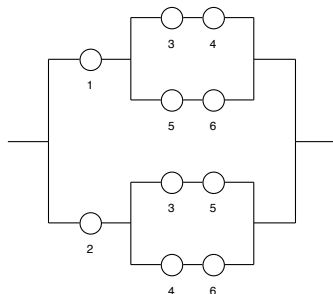
STK3405 – Lecture 22

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STK3405/4405 - 2019, problem 1

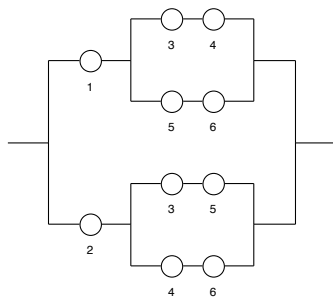


Let (C, ϕ) be a binary monotone system with component set of the system is $C = \{1, 2, \dots, 6\}$. $\mathbf{X} = (X_1, X_2, \dots, X_6)$ is the vector of component state variables, where $X_1, X_2, \dots, X_6$ are stochastically independent.

Let $\mathbf{p} = (p_1, p_2, \dots, p_6)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, \dots, 6$.



STK3405/4405 - 2019, problem 1a



SOLUTION:

Minimal path sets: $P_1 = \{1, 3, 4\}$, $P_2 = \{1, 5, 6\}$, $P_3 = \{2, 3, 5\}$, $P_4 = \{2, 4, 6\}$.

Minimal cut sets: $K_1 = \{1, 2\}$, $K_2 = \{3, 6\}$, $K_3 = \{4, 5\}$, $K_4 = \{1, 3, 4\}$,
 $K_5 = \{1, 5, 6\}$, $K_6 = \{2, 3, 5\}$, $K_7 = \{2, 4, 6\}$



(b) We let $h(\mathbf{p}) = P(\phi = 1)$ denote the reliability function of the system. Show that:

$$h(1_1, 1_2, \mathbf{p}) = E[\phi(1_1, 1_2, \mathbf{X})] = (p_3 \amalg p_6) \cdot (p_4 \amalg p_5),$$

$$h(1_1, 0_2, \mathbf{p}) = E[\phi(1_1, 0_2, \mathbf{X})] = (p_3 \cdot p_4) \amalg (p_5 \cdot p_6),$$

$$h(0_1, 1_2, \mathbf{p}) = E[\phi(0_1, 1_2, \mathbf{X})] = (p_3 \cdot p_5) \amalg (p_4 \cdot p_6),$$

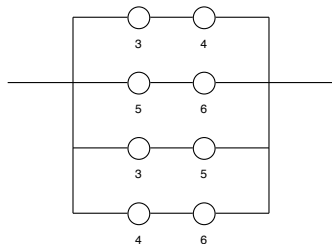
and use this to find $h(\mathbf{p})$.



STK3405/4405 - 2019, problem 1b (cont.)

SOLUTION:

CASE 1. $X_1 = 1$ and $X_2 = 1$.



Minimal path sets: $P'_1 = \{3, 4\}$, $P'_2 = \{5, 6\}$, $P'_3 = \{3, 5\}$, $P'_4 = \{4, 6\}$.

Minimal cut sets: $K_2 = \{3, 6\}$, $K_3 = \{4, 5\}$



STK3405/4405 - 2019, problem 1b (cont.)

We recall that the minimal path sets of the system are:

$$P_1 = \{1, 3, 4\}, P_2 = \{1, 5, 6\}, P_3 = \{2, 3, 5\}, P_4 = \{2, 4, 6\}.$$

In this case the resulting minimal path sets, where we have removed **component 1** and **component 2** which we know are functioning, are:

$$P'_1 = \{3, 4\}, P'_2 = \{5, 6\}, P'_3 = \{3, 5\}, P'_4 = \{4, 6\}.$$

This is a series connection of two parallel connections, (3 and 6 in parallel and 4 and 5 in parallel), which implies that:

$$\phi(1_1, 1_2, \mathbf{X}) = (X_3 \text{ II } X_6) \cdot (X_4 \text{ II } X_5),$$

and hence:

$$h(1_1, 1_2, \mathbf{p}) = (p_3 \text{ II } p_6) \cdot (p_4 \text{ II } p_5).$$



STK3405/4405 - 2019, problem 1b (cont.)

Alternatively, we recall that the minimal cut sets of the system are:

$$K_1 = \{1, 2\}, K_2 = \{3, 6\}, K_3 = \{4, 5\}, K_4 = \{1, 3, 4\}, \\ K_5 = \{1, 5, 6\}, K_6 = \{2, 3, 5\}, K_7 = \{2, 4, 6\}$$

In this case the minimal cut sets containing **components 1 or 2** can be removed. The remaining minimal cut sets are:

$$K_2 = \{3, 6\} \text{ and } K_3 = \{4, 5\}.$$

This is again a series connection of two parallel connections, (3 and 6 in parallel and 4 and 5 in parallel), which implies that:

$$\phi(1_1, 1_2, \mathbf{X}) = (X_3 \amalg X_6) \cdot (X_4 \amalg X_5),$$

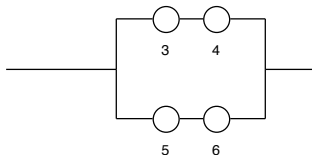
and hence:

$$h(1_1, 1_2, \mathbf{p}) = (p_3 \amalg p_6) \cdot (p_4 \amalg p_5).$$



STK3405/4405 - 2019, problem 1b (cont.)

CASE 2. $X_1 = 1$ and $X_2 = 0$.



Minimal path sets: $P'_1 = \{3, 4\}$, $P'_2 = \{5, 6\}$.

Minimal cut sets: $K_2 = \{3, 6\}$, $K_3 = \{4, 5\}$, $K'_6 = \{3, 5\}$, $K'_7 = \{4, 6\}$



STK3405/4405 - 2019, problem 1b (cont.)

We recall that the minimal path sets of the system are:

$$P_1 = \{1, 3, 4\}, P_2 = \{1, 5, 6\}, P_3 = \{2, 3, 5\}, P_4 = \{2, 4, 6\}.$$

In this case the minimal paths containing **component 2** can be removed, and thus, the resulting minimal path sets, where we have removed **component 1** which we know is functioning, are:

$$P'_1 = \{3, 4\} \text{ and } P'_2 = \{5, 6\}.$$

This is a parallel connection of two series connections, (3 and 4 in series and 5 and 6 in series), which implies that:

$$\phi(1_1, 0_2, \mathbf{X}) = (X_3 \cdot X_4) \amalg (X_5 \cdot X_6),$$

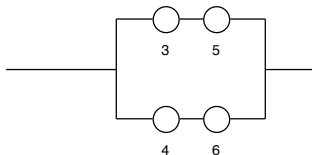
and hence:

$$h(1_1, 0_2, \mathbf{p}) = (p_3 \cdot p_4) \amalg (p_5 \cdot p_6).$$



STK3405/4405 - 2019, problem 1b (cont.)

CASE 3. $X_1 = 0$ and $X_2 = 1$.



Minimal path sets: $P'_3 = \{3, 5\}$ and $P'_4 = \{4, 6\}$.

Minimal cut sets: $K_2 = \{3, 6\}$, $K_3 = \{4, 5\}$, $K'_4 = \{3, 4\}$, $K'_5 = \{5, 6\}$



STK3405/4405 - 2019, problem 1b (cont.)

We recall that the minimal path sets of the system are:

$$P_1 = \{1, 3, 4\}, P_2 = \{1, 5, 6\}, P_3 = \{2, 3, 5\}, P_4 = \{2, 4, 6\}.$$

In this case the minimal paths containing **component 1** can be removed, and thus, the resulting minimal path sets, where we have removed **component 2** which we know is functioning, are:

$$P'_3 = \{3, 5\} \text{ and } P'_4 = \{4, 6\}.$$

This is a parallel connection of two series connections, (3 and 5 in series and 4 and 6 in series), which implies that:

$$\phi(0_1, 1_2, \mathbf{X}) = (X_3 \cdot X_5) \amalg (X_4 \cdot X_6),$$

and hence:

$$h(0_1, 1_2, \mathbf{p}) = (p_3 \cdot p_5) \amalg (p_4 \cdot p_6).$$



STK3405/4405 - 2019, problem 1b (cont.)

Finally, since $\{1, 2\}$ is a minimal cut set, we have:

$$h(0_1, 0_2, \mathbf{p}) = \phi(0_1, 0_2, \mathbf{X}) = 0$$

Hence, combining all these results we get:

$$\begin{aligned} h(\mathbf{p}) &= p_1 p_2 h(1_1, 1_2, \mathbf{p}) + p_1(1 - p_2)h(1_1, 0_2, \mathbf{p}) + (1 - p_1)p_2 h(0_1, 1_2, \mathbf{p}) \\ &= p_1 p_2 [(p_3 \amalg p_6) \cdot (p_4 \amalg p_5)] \\ &\quad + p_1(1 - p_2)[(p_3 \cdot p_4) \amalg (p_5 \cdot p_6)] \\ &\quad + (1 - p_1)p_2[(p_3 \cdot p_5) \amalg (p_4 \cdot p_6)] \end{aligned}$$



STK3405/4405 - 2019, problem 1c

In the remaining part of this problem we assume that all components have equal reliability p , i.e., $p_1 = \dots = p_6 = p$. The reliability function can then be written as $h(p)$ instead of $h(\mathbf{p})$.

(c) Use the results from (b) to show that:

$$h(p) = p^4 \cdot (2 - p)^2 + 2p^3(1 - p)(2 - p^2)$$

In particular, show that:

$$h\left(\frac{1}{2}\right) = 23 \cdot \left(\frac{1}{2}\right)^6$$

SOLUTION: We start by noting that:

$$s \text{ II } s = 1 - (1 - s)(1 - s) = 2s - s^2, \quad \text{for all } s.$$



STK3405/4405 - 2019, problem 1c (cont.)

Inserting $p_1 = \dots = p_6 = p$ into $h(\mathbf{p})$ we get:

$$\begin{aligned}h(p) &= p_1 p_2 [(p_3 \amalg p_6) \cdot (p_4 \amalg p_5)] \\&\quad + p_1 (1 - p_2) [(p_3 \cdot p_4) \amalg (p_5 \cdot p_6)] + (1 - p_1) p_2 [(p_3 \cdot p_5) \amalg (p_4 \cdot p_6)] \\&= p^2 [(2p - p^2) \cdot (2p - p^2)] + p(1 - p)[2p^2 - p^4] + (1 - p)p[2p^2 - p^4] \\&= p^2 \cdot (2p - p^2)^2 + 2p(1 - p) \cdot (2p^2 - p^4) \\&= p^4 \cdot (2 - p)^2 + 2p^3(1 - p)(2 - p^2).\end{aligned}$$

In particular we have:

$$\begin{aligned}h\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^4 \cdot \left(\frac{3}{2}\right)^2 + 2 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{7}{4}\right) \\&= [3^2 + 2 \cdot 7] \cdot \left(\frac{1}{2}\right)^6 = 23 \cdot \left(\frac{1}{2}\right)^6.\end{aligned}$$



(d) Let $S = \sum_{i=1}^6 X_i$. Explain why the distribution of S is given by:

$$P(S = s) = \binom{6}{s} p^s (1 - p)^{6-s}, \quad s = 0, 1, \dots, 6.$$

SOLUTION:

The random variable S is the sum of the independent and identically binary variables $X_1, \dots, X_6$.

Hence, $S \sim \text{Bin}(6, p)$, and thus $P(S = s)$ can be expressed as claimed.



(e) Show that:

$$h(p) = \sum_{s=0}^6 b_s p^s (1-p)^{6-s}$$

where b_s denotes the number of path sets (minimal and non-minimal) having exactly s components, $s = 0, 1, \dots, 6$.

SOLUTION:

We start by noting that:

$$b_s = \sum_{\{\mathbf{x} : \sum_{i=1}^6 x_i = s\}} \phi(\mathbf{x}), \quad s = 0, 1, \dots, 6.$$



STK3405/4405 - 2019, problem 1e (cont.)

Moreover, the conditional distribution of $\mathbf{X}$ given S is:

$$P(\mathbf{X} = \mathbf{x} \mid S = s) = \frac{p^{\sum_{i=1}^6 x_i} (1-p)^{6-\sum_{i=1}^6 x_i}}{\binom{6}{s} p^s (1-p)^{6-s}} = \frac{1}{\binom{6}{s}},$$

for all $\mathbf{x}$ such that $\sum_{i=1}^6 x_i = s$, and zero otherwise.

From this it follows that:

$$\begin{aligned} E[\phi(\mathbf{X}) \mid S = s] &= \sum_{\{\mathbf{x} : \sum_{i=1}^6 x_i = s\}} \phi(\mathbf{x}) P(\mathbf{X} = \mathbf{x} \mid S = s) \\ &= \frac{1}{\binom{6}{s}} \sum_{\{\mathbf{x} : \sum_{i=1}^6 x_i = s\}} \phi(\mathbf{x}) = \frac{b_s}{\binom{6}{s}} \end{aligned}$$



Finally, the system reliability, h , expressed as a function of p , is given by:

$$\begin{aligned} h(p) &= E[\phi(\mathbf{X})] = \sum_{s=0}^6 E[\phi(\mathbf{X})|S=s]P(S=s) \\ &= \sum_{s=0}^6 \frac{b_s}{\binom{6}{s}} \binom{6}{s} p^s (1-p)^{6-s} = \sum_{s=0}^6 b_s p^s (1-p)^{6-s}. \end{aligned}$$



STK3405/4405 - 2019, problem 1f

(f) Show that:

$$\sum_{s=0}^6 b_s = 23.$$

SOLUTION: By inserting $p = \frac{1}{2}$ into the expression for $h(p)$ we get:

$$\begin{aligned} h\left(\frac{1}{2}\right) &= \sum_{s=0}^6 b_s \left(\frac{1}{2}\right)^s \left(1 - \left(\frac{1}{2}\right)\right)^{6-s} \\ &= \sum_{s=0}^6 b_s \left(\frac{1}{2}\right)^6 = \left[\sum_{s=0}^6 b_s\right] \left(\frac{1}{2}\right)^6 = 23 \cdot \left(\frac{1}{2}\right)^6, \end{aligned}$$

where the last equality follows by the last result in (c). Hence:

$$\sum_{s=0}^6 b_s = 23.$$



STK3405/4405 - 2019, problem 1g

(g) Determine $b_0, b_1, \dots, b_6$.

SOLUTION: Since the smallest path sets have 3 components, we must have:

$$b_0 = b_1 = b_2 = 0.$$

We know from (a) that there are 4 minimal paths, all of size 3. Hence we have:

$$b_3 = 4.$$

Since all cut sets have at least 2 components, all sets of size 5 or 6 must be path sets. Hence, we have:

$$b_5 = \binom{6}{5} = 6, \quad \text{and} \quad b_6 = \binom{6}{6} = 1.$$



STK3405/4405 - 2019, problem 1g (cont.)

In order to determine b_4 , we could go through all sets of size 4, i.e., $\binom{6}{s} = 15$ sets, and count the path sets among these sets.

Alternatively, we can apply the result from (f). This gives us an equation which we can use to determine b_4 :

$$0 + 0 + 0 + 4 + b_4 + 6 + 1 = 23$$

This implies that:

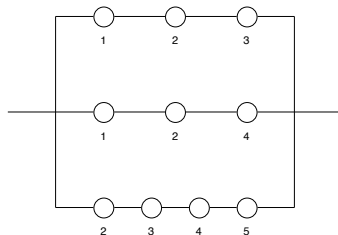
$$b_4 = 12.$$

NOTE: By the same arguments as we have used in this problem we can show more generally that if (C, ϕ) is a binary monotone system of order n , then:

$$\sum_{s=0}^n b_s = h\left(\frac{1}{2}\right) \cdot 2^n.$$



STK3405/4405 - 2021, problem 1

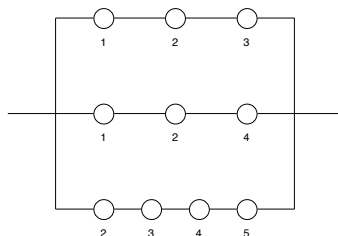


Binary monotone system (C, ϕ) with component set of the system is $C = \{1, \dots, 5\}$. $\mathbf{X} = (X_1, \dots, X_5)$ is the vector of component state variables, where $X_1, \dots, X_5$ are stochastically independent.

Let $\mathbf{p} = (p_1, \dots, p_5)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, \dots, 5$.



STK3405/4405 - 2021, problem 1a



(a) Find the minimal path sets (3 sets) and the minimal cut sets (5 sets) of the system.

SOLUTION:

Minimal path sets: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{2, 3, 4, 5\}$

Minimal cut sets: $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2\}$, $\{3, 4\}$



STK3405/4405 - 2021, problem 1b

(b) We let $h(\mathbf{p}) = P(\phi = 1)$ denote the reliability function of the system. Show that:

$$h(\mathbf{p}) = p_2 \cdot [p_1 \cdot (p_3 \text{ II } p_4) + (1 - p_1) \cdot p_3 p_4 p_5]$$

SOLUTION: We note that component 2 is in series with the rest of the system. Hence, $h(0_2, \mathbf{p}) = 0$, and we get:

$$h(\mathbf{p}) = p_2 \cdot h(1_2, \mathbf{p}) + (1 - p_2)h(0_2, \mathbf{p}) = p_2 \cdot h(1_2, \mathbf{p})$$

In order to find $h(1_2, \mathbf{p})$ we do a pivotal decomposition with respect to component 1. If component 1 is functioning, the rest of the system is a parallel connection of components 3 and 4, while if 1 is failed, the rest of the system is a series connection of components 3, 4 and 5. Hence, we get:

$$\begin{aligned} h(\mathbf{p}) &= p_2 \cdot h(1_2, \mathbf{p}) \\ &= p_2 \cdot [p_1 \cdot h(1_1, 1_2, \mathbf{p}) + (1 - p_1) \cdot h(0_1, 1_2, \mathbf{p})] \\ &= p_2 \cdot [p_1 \cdot (p_3 \text{ II } p_4) + (1 - p_1) \cdot p_3 p_4 p_5] \end{aligned}$$



STK3405/4405 - 2021, problem 1c

The Birnbaum measure for the **reliability importance** of component i is defined as:

$$I_B^{(i)} = P(\text{Component } i \text{ is critical for the system}), \quad i = 1, 2, \dots, 5.$$

Show that:

$$I_B^{(i)} = \frac{\partial h(\mathbf{p})}{\partial p_i}, \quad i = 1, 2, \dots, 5.$$

SOLUTION: Component i is **critical** for the system if and only if:

$$\phi(1_i, \mathbf{X}) = 1, \text{ and } \phi(0_i, \mathbf{X}) = 0 \quad (1)$$

Since ϕ is non-decreasing in each argument, we always have that: $\phi(1_i, \mathbf{X}) \geq \phi(0_i, \mathbf{X})$. Thus, the condition (1) is equivalent to:

$$\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X}) = 1$$

(2)

STK3405/4405 - 2021, problem 1c (cont.)

Hence, since the component state variables are assumed to be independent, it follows for $i = 1, 2, \dots, 5$ that:

$$\begin{aligned} I_B^{(i)} &= P(\text{Component } i \text{ is critical for the system}) \\ &= P(\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X}) = 1) \\ &= E[\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X})] \\ &= E[\phi(1_i, \mathbf{X})] - E[\phi(0_i, \mathbf{X})] \\ &= h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}) \\ &= \frac{\partial}{\partial p_i} [p_i \cdot h(1_i, \mathbf{p}) + (1 - p_i) \cdot h(0_i, \mathbf{p})] \\ &= \frac{\partial h(\mathbf{p})}{\partial p_i} \quad \blacksquare \end{aligned}$$



STK3405/4405 - 2021, problem 1d

(d) Show that:

$$I_B^{(2)} = p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5$$

$$I_B^{(5)} = (1 - p_1) \cdot p_2 p_3 p_4$$

SOLUTION: By using the result from (c) we get:

$$I_B^{(2)} = \frac{\partial}{\partial p_2} [p_2 \cdot [p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5]]$$

$$= p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5$$

$$I_B^{(5)} = \frac{\partial}{\partial p_5} [p_2 \cdot [p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5]]$$

$$= (1 - p_1) \cdot p_2 p_3 p_4 \quad \blacksquare$$



STK3405/4405 - 2021, problem 1e

In the remaining part of this problem we assume that $0 < p_i < 1$, $i = 1, 2, \dots, 5$.

(e) Show that if $p_5 \geq p_2$, then $I_B^{(2)} > I_B^{(5)}$.

SOLUTION: In order to compare $I_B^{(2)}$ and $I_B^{(5)}$, we consider:

$$\begin{aligned} I_B^{(2)} - I_B^{(5)} &= p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5 - (1 - p_1) \cdot p_2 p_3 p_4 \\ &= p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 (p_5 - p_2) \end{aligned}$$

If $p_5 \geq p_2$, we observe that both terms in the difference between $I_B^{(2)}$ and $I_B^{(5)}$ are non-negative. Moreover, since we have assumed that $0 < p_i < 1$ for all i , the first term is strictly positive. Hence, we conclude $I_B^{(2)} > I_B^{(5)}$ ■



STK3405/4405 - 2021, problem 1f

(f) Show that if $p_1 \geq \frac{1}{2}$, then $I_B^{(2)} > I_B^{(5)}$.

SOLUTION: In order to compare $I_B^{(2)}$ and $I_B^{(5)}$, we again consider:

$$I_B^{(2)} - I_B^{(5)} = p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 (p_5 - p_2)$$

Since we have assumed that $0 < p_i < 1$ for all i , it follows that $(1 - p_1) \cdot p_3 p_4 > 0$ and that $(p_5 - p_2) > -1$. Hence, when $p_1 \geq \frac{1}{2}$, we get that:

$$\begin{aligned} I_B^{(2)} - I_B^{(5)} &> p_1 \cdot (p_3 \amalg p_4) - (1 - p_1) \cdot p_3 p_4 \\ &\geq \frac{1}{2} \cdot (p_3 \amalg p_4) - \frac{1}{2} \cdot p_3 p_4 \\ &= \frac{1}{2} (p_3 + p_4 - p_3 p_4 - p_3 p_4) \\ &= \frac{1}{2} [p_3(1 - p_4) + p_4(1 - p_3)] > 0 \end{aligned}$$

where the last inequality again follows since $0 < p_i < 1$ for all i . From this we conclude that $I_B^{(2)} > I_B^{(5)}$ ■



STK3405/4405 - 2021, problem 1g

(g) In this point we assume more specifically that $p_1 = p_5 = \frac{1}{10}$ and that $p_2 = p_3 = p_4 = \frac{9}{10}$. Calculate $I_B^{(2)}$ and $I_B^{(5)}$ and compare the results. Comment your findings.

SOLUTION: By using the result from (d) we get:

$$\begin{aligned} I_B^{(2)} &= p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5 \\ &= \frac{1}{10} \cdot \left(\frac{9}{10} + \frac{9}{10} - \frac{9}{10} \frac{9}{10} \right) + \frac{9}{10} \cdot \frac{9}{10} \frac{9}{10} \frac{1}{10} \\ &= \frac{990}{10000} + \frac{729}{10000} = \frac{1719}{10000} \end{aligned}$$

$$\begin{aligned} I_B^{(5)} &= (1 - p_1) \cdot p_2 p_3 p_4 \\ &= \frac{9}{10} \cdot \frac{9}{10} \frac{9}{10} \frac{9}{10} = \frac{6561}{10000} \end{aligned}$$



STK3405/4405 - 2021, problem 1g (cont.)

Thus, in this case we have $I_B^{(2)} < I_B^{(5)}$ which is the opposite ranking compared to the cases considered in the two previous points. We note that with these component reliabilities we have:

$p_1 < \frac{1}{2}$ (Thus, the result from (f) does not apply in this case)

$p_5 < p_2$ (Thus, the result from (e) does not apply in this case)

Indeed when p_1 is small, then the system is, with a high probability, reduced to a *series connection* of the components 2, 3, 4, 5. In a series system the most important component is the one with the smallest reliability, i.e., component 5 in this case ■



STK3405/4405 - 2021, problem 1h

The Birnbaum measure for the *structural importance* of component i is defined as:

$$J_B^{(i)} = \frac{1}{2^{5-1}} \sum_{(\cdot, i, \mathbf{x})} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})], \quad i = 1, 2, \dots, 5.$$

(h) Explain briefly why: $J_B^{(2)} > J_B^{(i)}$ for $i = 1, 3, 4, 5$.

SOLUTION: We observe that component 2 is in **series with the rest of the system**, while none of the other components are in series with the rest of the system.

From this it follows that the structural importance of component 2 is greater than the structural importance of any of the other components. See Exercise 5.3 in the textbook.

On the next slide we include a formal proof of this result.



STK3405/4405 - 2021, problem 1h (cont.)

Since component 2 is in series with the rest of the system, we have:

$$\phi(0_2, \mathbf{x}) = 0, \quad \text{for all } (\cdot_2, \mathbf{x}) \in \{0, 1\}^4$$

We then choose another component $j \neq 2$. Since j is *not* in series with the rest of the system, we have:

$$\phi(0_j, \mathbf{x}) = 1, \quad \text{for at least one } (\cdot_j, \mathbf{x}) \in \{0, 1\}^4$$

Hence, we then get:

$$\begin{aligned} 2^4 J_B^{(2)} &= \sum_{(\cdot_2, \mathbf{x})} [\phi(1_2, \mathbf{x}) - \phi(0_2, \mathbf{x})] = \sum_{(\cdot_2, \mathbf{x})} [\phi(1_2, \mathbf{x}) + \phi(0_2, \mathbf{x})] \\ &= \sum_{\mathbf{x}} \phi(\mathbf{x}) = \sum_{(\cdot_j, \mathbf{x})} [\phi(1_j, \mathbf{x}) + \phi(0_j, \mathbf{x})] \\ &> \sum_{(\cdot_j, \mathbf{x})} [\phi(1_j, \mathbf{x}) - \phi(0_j, \mathbf{x})] = 2^4 J_B^{(j)} \quad \blacksquare \end{aligned}$$

