STK3405 – Lecture 11

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Chapter 5

Structural and reliability importance for components in binary monotone systems



Importance measures

- A measure of importance can be used to identify components that should be improved in order to increase the system reliability.
- A measure of importance can be used to identify components that most likely have failed, given that the system has failed.



Section 5.1

Structural importance of a component



Criticality

Definition (Criticality)

Let (C, ϕ) be a binary monotone system, and let $i \in C$. We say that component i is critical for the system if:

$$\phi(1_i, \boldsymbol{x}) = 1 \text{ and } \phi(0_i, \boldsymbol{x}) = 0.$$

If this is the case, we also say that $(\cdot_i, \boldsymbol{x})$ is a critical vector for component i.

NOTE: Criticality is strongly related to the notion of relevance: A component i in a binary monotone system (C, ϕ) is relevant if and only if there exists at least one critical vector for i.



Criticality (cont.)

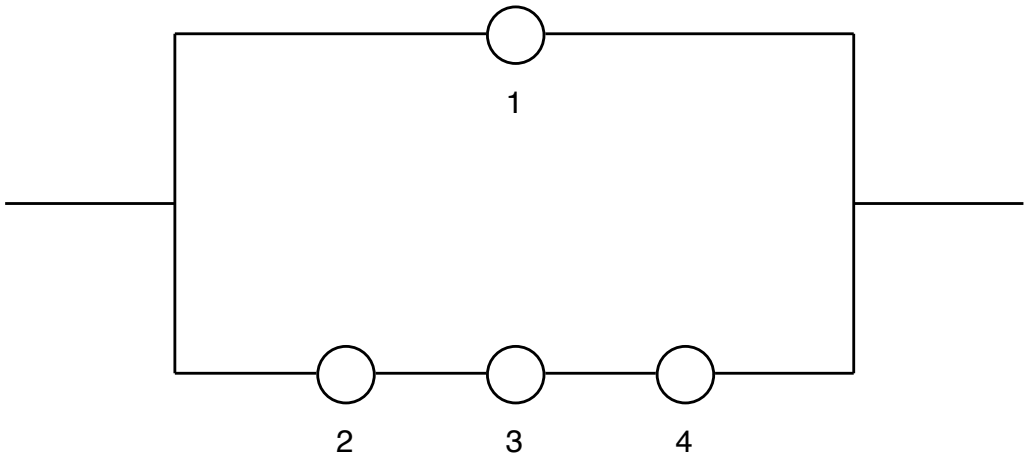


Figure: A binary monotone system (C, ϕ)

The structure function of the system (C, ϕ) is given by:

$$\phi(\boldsymbol{x}) = x_1 \amalg (x_2 \cdot x_3 \cdot x_4)$$



Criticality (cont.)

Component 1 is *critical* if $(\cdot_1, \boldsymbol{x})$ is:

$$(\cdot, 0, 0, 0), (\cdot, 1, 0, 0), (\cdot, 0, 1, 0), (\cdot, 0, 0, 1),$$

$$(\cdot, 1, 1, 0), (\cdot, 1, 0, 1), (\cdot, 0, 1, 1).$$

Component 2 is *critical* if $(\cdot_2, \boldsymbol{x}) = (0, \cdot, 1, 1)$,

Component 3 is *critical* if $(\cdot_3, \boldsymbol{x}) = (0, 1, \cdot, 1)$,

Component 4 is *critical* if $(\cdot_4, \boldsymbol{x}) = (0, 1, 1, \cdot)$.



Structural importance

Based on this Birnbaum suggested the following measure of structural importance of a component in a binary monotone system:

Definition (Structural importance)

Let (C, ϕ) be a binary monotone system of order n, and let $i \in C$. The Birnbaum measure for the structural importance of component i, denoted $J_B^{(i)}$, is defined as:

$$J_B^{(i)} = \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})].$$

Note that the denominator, 2^{n-1} is the total number of states for the $n-1$ other components. Thus, $J_B^{(i)}$ can be interpreted as the fraction of all states for the $n-1$ other components where component i is critical.



Structural importance

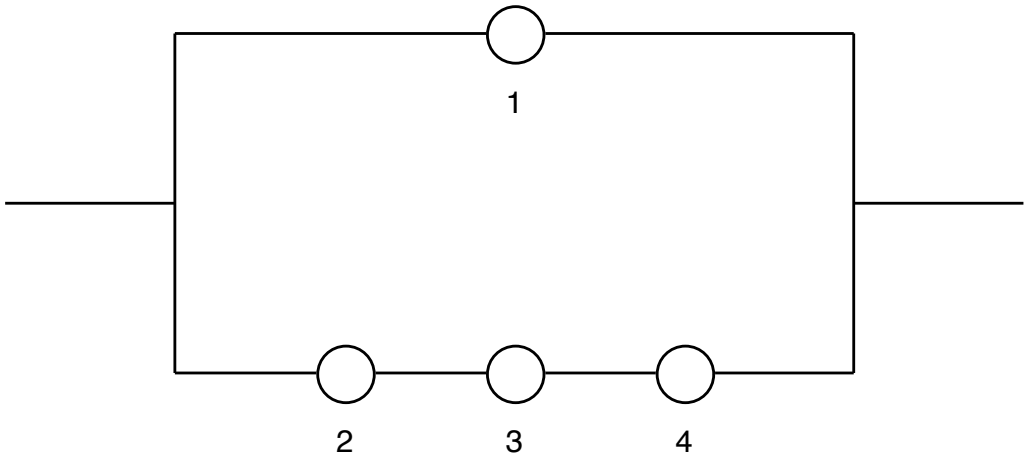


Figure: A binary monotone system (C, ϕ)

For this system we have the following structural importance measures:

$$J_B^{(1)} = \frac{7}{2^{4-1}} = \frac{7}{8}, \qquad J_B^{(2)} = J_B^{(3)} = J_B^{(4)} = \frac{1}{2^{4-1}} = \frac{1}{8}.$$



Structural importance

Let (C, ϕ) be a 2-out-of-3 system. To compute the structural importance of component 1, we note that the critical vectors for this component are $(\cdot, 1, 0)$ and $(\cdot, 0, 1)$. Hence, we have:

$$J_B^{(1)} = \frac{2}{2^{3-1}} = \frac{1}{2}.$$

By similar arguments, we find that:

$$J_B^{(2)} = J_B^{(3)} = \frac{1}{2}.$$

So in a 2-out-of-3 system, all of the components have the same structural importance. This is intuitively obvious since the structure function is symmetrical with respect to the components.



Section 5.2

Reliability importance of a component



Reliability importance of a component

Definition (Reliability importance of a component)

Let (C, ϕ) be a binary monotone system, and let $i \in C$. Moreover, let $\boldsymbol{X}$ be the vector of component state variables.

The Birnbaum measure for the reliability importance of component i, denoted $I_B^{(i)}$ is defined as:

$$\begin{aligned} I_B^{(i)} &= P(\text{Component } i \text{ is critical for the system}) \\ &= P(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1). \end{aligned}$$



Reliability importance of a component (cont.)

Since the difference $\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})$ is a binary variable, it follows that:

$$I_B^{(i)} = \mathrm{E}[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})] = \mathrm{E}[\phi(1_i, \boldsymbol{X})] - \mathrm{E}[\phi(0_i, \boldsymbol{X})].$$

In particular, if the component state variables of the system are independent, and $P(X_i = 1) = p_i$ for $i \in C$, we get that:

$$I_B^{(i)} = h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p}).$$



Reliability importance of a component (cont.)

Theorem (Partial derivative formula)

Let (C, ϕ) be a binary monotone system where the component state variables are independent, and $P(X_i = 1) = p_i$ for $i \in C$.

Then:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \qquad \text{for all } i \in C.$$

PROOF: By pivotal decomposition we have:

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p})$$

By differentiating this identity with respect to p_i we get:

$$\frac{\partial h(\boldsymbol{p})}{\partial p_i} = h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p}).$$

Hence, the result follows.



Reliability importance inequalities

Theorem (Reliability importance inequalities)

For a binary monotone system, (C, ϕ), we always have

$$0 \leq I_B^{(i)} \leq 1.$$

Assume that the component state variables are independent, and $P(X_j = 1) = p_j$, where $0 < p_j < 1$ for all $j \in C$.

If component i is relevant, we have:

$$0 < I_B^{(i)}.$$

Furthermore, if there exists at least one other relevant component, we also have:

$$I_B^{(i)} < 1.$$



Reliability importance inequalities (cont.)

PROOF: We note that the first inequality follows directly from the definition since the reliability importance is a *probability*.

We then assume that the component state variables are independent, and that $P(X_j = 1) = p_j$, where $0 < p_j < 1$ for all $j \in C$.

If component i is relevant, we know that h is strictly increasing in p_i.

That is, we must have:

$$\frac{\partial h(\boldsymbol{p})}{\partial p_i} > 0.$$

Combining this with the partial derivative formula, we get that $0 < I_B^{(i)}$.



Reliability importance inequalities (cont.)

Finally, we assume that there exists at least one other relevant component $k \in C$.

To show that this implies that $I_B^{(i)} < 1$, we assume instead that $I_B^{(i)} = 1$, and show that this leads to a contradiction.

By this assumption, it follows that :

$$P(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1) = 1$$

Since $0 < p_j < 1$, for all $j \in C$, it follows that $P((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})) > 0$ for all $(\cdot_i, \boldsymbol{x})$.

Hence, we must have that:

$$\phi(1_i, \boldsymbol{x}) = 1 \text{ and } \phi(0_i, \boldsymbol{x}) = 0 \text{ for all } (\cdot_i, \boldsymbol{x}).$$



Reliability importance inequalities (cont.)

At the same time, since component k is relevant, there exists a vector $(\cdot_k, \boldsymbol{y})$ such that:

$$\phi(1_k, \boldsymbol{y}) = 1 \text{ and } \phi(0_k, \boldsymbol{y}) = 0.$$

If $y_i = 1$, it follows that $\phi(1_i, 0_k, \boldsymbol{y}) = 0$, contradicting that $\phi(1_i, \boldsymbol{x}) = 1$ for all $(\cdot_i, \boldsymbol{x})$.

If $y_i = 0$, it follows that $\phi(0_i, 1_k, \boldsymbol{y}) = 1$, contradicting that $\phi(0_i, \boldsymbol{x}) = 0$ for all $(\cdot_i, \boldsymbol{x})$.

Hence, we conclude that for both possible values of y_i we end up with contradictions.

Thus, the only possibility is that $I_B^{(i)} < 1$.



Reliability importance and structural importance

Theorem (Reliability importance and structural importance)

Consider a binary monotone system (C, ϕ) where the component state variables are independent, and where $P(X_i = 1) = \frac{1}{2}$ for all $i \in C$. Then we have:

$$I_B^{(i)} = J_B^{(i)}$$

PROOF: If the component state variables are independent, and $P(X_i = 1) = \frac{1}{2}$ for all $i \in C$, we have:

$$P((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})) = \prod_{j \neq i} P(X_j = x_j) = \prod_{j \neq i} (\frac{1}{2}) = \frac{1}{2^{n-1}}.$$

From this the result follows.



Reliability importance examples

In the following examples we consider binary monotone systems (C, ϕ) where $C = \{1, \ldots, n\}$.

We also assume that the component state variables are independent, and that:

$$P(X_i = 1) = p_i, \quad i \in C.$$

Without loss of generality we assume that the components are ordered so that:

$$p_1 \leq p_2 \leq \ldots \leq p_n. \tag{1}$$



Reliability importance examples (cont.)

Let (C, ϕ) be a series system. Then for all $i \in C$ we have:

$$I_B^{(i)} = \frac{\partial \prod_{j=1}^{n} p_j}{\partial p_i} = \prod_{j \neq i} p_j.$$

Hence, by the ordering (1), we get that:

$$I_B^{(1)} \geq I_B^{(2)} \geq \cdots \geq I_B^{(n)}.$$

Thus, in a series system the *worst* component, i.e., the one with the smallest reliability, has the greatest reliability importance.



Reliability importance examples (cont.)

Let (C, ϕ) be a parallel system. Then for all $i \in C$ we have:

$$I_B^{(i)} = \frac{\partial \coprod_{j=1}^n p_j}{\partial p_i} = \frac{\partial[1 - \prod_{j=1}^n (1 - p_j)]}{\partial p_i} = \prod_{j \neq i} (1 - p_j).$$

Hence, from the ordering (1)

$$I_B^{(1)} \leq I_B^{(2)} \leq \cdots \leq I_B^{(n)}.$$

Thus, in a parallel system the *best* component, i.e., the one with the greatest reliability, has the greatest reliability importance.



Reliability importance examples (cont.)

Let (C, ϕ) be a 2-out-of-3 system. It is then easy to show that:

$$\phi(\boldsymbol{X}) = X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3.$$

Hence, we have:

$$h(\boldsymbol{p}) = p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3.$$

This implies that:

$$I_B^{(1)} = \frac{\partial h(\boldsymbol{p})}{\partial p_1} = p_2 + p_3 - 2p_2p_3,$$

$$I_B^{(2)} = \frac{\partial h(\boldsymbol{p})}{\partial p_2} = p_1 + p_3 - 2p_1p_3,$$

$$I_B^{(3)} = \frac{\partial h(\boldsymbol{p})}{\partial p_3} = p_1 + p_2 - 2p_1p_2.$$



Reliability importance examples (cont.)

We then consider the function $f(p,q) = p + q - 2pq$ and note that:

$$I_B^{(1)} = f(p_2, p_3), \quad I_B^{(2)} = f(p_1, p_3), \quad I_B^{(3)} = f(p_1, p_2).$$

Moreover, the partial derivatives of f are respectively:

$$\frac{\partial f}{\partial p} = 1 - 2q, \qquad \frac{\partial f}{\partial q} = 1 - 2p.$$

If $p, q \leq \frac{1}{2}$, f is non-decreasing in p and q. Thus, if $p_1 \leq p_2 \leq p_3 \leq \frac{1}{2}$, we have:

$$f(p_1, p_2) \leq f(p_1, p_3) \leq f(p_2, p_3).$$

Hence, in this case we have:

$$I_B^{(3)} \leq I_B^{(2)} \leq I_B^{(1)}. \tag{2}$$



Reliability importance examples (cont.)

If $p, q \geq \frac{1}{2}$, f is non-increasing in p and q. Thus, if $\frac{1}{2} \leq p_1 \leq p_2 \leq p_3$, we have:

$$f(p_2, p_3) \leq f(p_1, p_3) \leq f(p_1, p_2).$$

Hence, in this case we have:

$$I_B^{(1)} \leq I_B^{(2)} \leq I_B^{(3)}. \tag{3}$$



Reliability importance examples (cont.)

If $p_1 = \frac{1}{2} - z$, $p_2 = \frac{1}{2}$ and $p_3 = \frac{1}{2} + z$, where $z \in (0, \frac{1}{2})$, we get:

$$I_B^{(1)} = (\frac{1}{2}) + (\frac{1}{2} + z) - 2 \cdot (\frac{1}{2})(\frac{1}{2} + z) = \frac{1}{2},$$
$$I_B^{(2)} = (\frac{1}{2} - z) + (\frac{1}{2} + z) - 2 \cdot (\frac{1}{2} - z)(\frac{1}{2} + z) = \frac{1}{2} + 2z^2,$$
$$I_B^{(3)} = (\frac{1}{2} - z) + (\frac{1}{2}) - 2 \cdot (\frac{1}{2} - z)(\frac{1}{2}) = \frac{1}{2},$$

Hence in this case we have:

$$I_B^{(1)} = I_B^{(3)} \leq I_B^{(2)}. \tag{4}$$

Note that this result holds also if $z \in (-\frac{1}{2}, 0)$ in which case $p_1 > p_2 > p_3$.



Reliability importance examples (cont.)

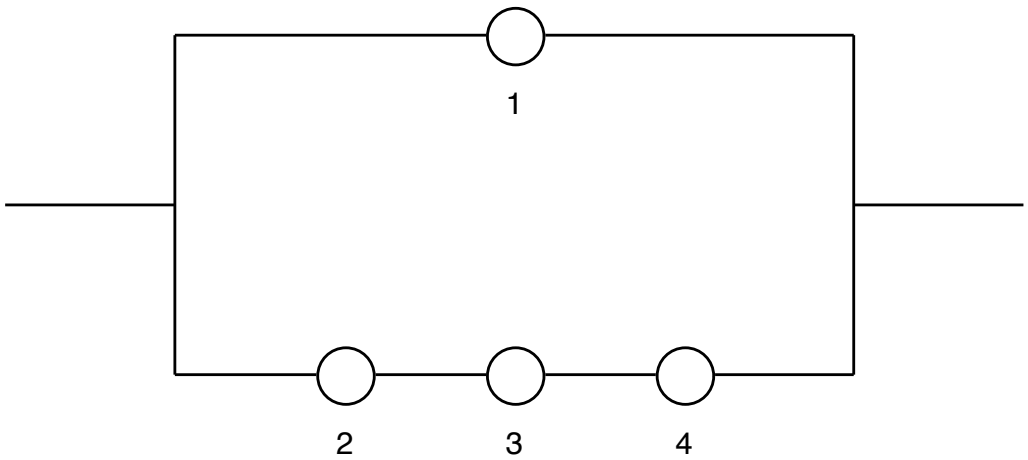


Figure: A binary monotone system (C, ϕ)

The structure function of this system is:

$$\phi(\boldsymbol{X}) = X_1 \amalg (X_2 \cdot X_3 \cdot X_4) = X_1 + X_2 \cdot X_3 \cdot X_4 - X_1 \cdot X_2 \cdot X_3 \cdot X_4$$

Thus, the reliability function is given by:

$$h(\boldsymbol{p}) = p_1 + p_2 \cdot p_3 \cdot p_4 - p_1 \cdot p_2 \cdot p_3 \cdot p_4$$



Reliability importance examples (cont.)

Hence we have:

$$I_B^{(1)} = 1 - p_2 \cdot p_3 \cdot p_4$$

$$I_B^{(2)} = p_3 \cdot p_4 - p_1 \cdot p_3 \cdot p_4 = (1 - p_1) \cdot p_3 \cdot p_4$$

$$I_B^{(3)} = p_2 \cdot p_4 - p_1 \cdot p_2 \cdot p_4 = (1 - p_1) \cdot p_2 \cdot p_4$$

$$I_B^{(4)} = p_2 \cdot p_3 - p_1 \cdot p_2 \cdot p_3 = (1 - p_1) \cdot p_2 \cdot p_3$$

If $p_1 = p_2 = p_3 = p_4 = p \in (0, 1)$, we have:

$$I_B^{(1)} = 1 - p^3$$

$$I_B^{(i)} = p^2 - p^3 < I_B^{(1)}, \quad i = 2, 3, 4.$$



Reliability importance examples (cont.)

Assume instead that $p_1 = 0.1$ and that $p_2 = p_3 = p_4 = 0.9$. Then we get:

$$I_B^{(1)} = 1 - p_2 \cdot p_3 \cdot p_4 = 1 - 0.9^3 = 0.271$$

$$I_B^{(2)} = p_3 \cdot p_4 - p_1 \cdot p_3 \cdot p_4 = (1 - p_1) \cdot p_3 \cdot p_4 = 0.9^3 = 0.729$$

$$I_B^{(3)} = p_2 \cdot p_4 - p_1 \cdot p_2 \cdot p_4 = (1 - p_1) \cdot p_2 \cdot p_4 = 0.9^3 = 0.729$$

$$I_B^{(4)} = p_2 \cdot p_3 - p_1 \cdot p_2 \cdot p_3 = (1 - p_1) \cdot p_2 \cdot p_3 = 0.9^3 = 0.729$$

Thus, in this case we have:

$$I_B^{(1)} < I_B^{(2)} = I_B^{(3)} = I_B^{(4)}.$$

