STK3405 – Lecture 13

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Exercise 4.13

Construct a reduced ordered binary decision diagram of a 3-out-5-system.



Exercise 4.13 (cont.)

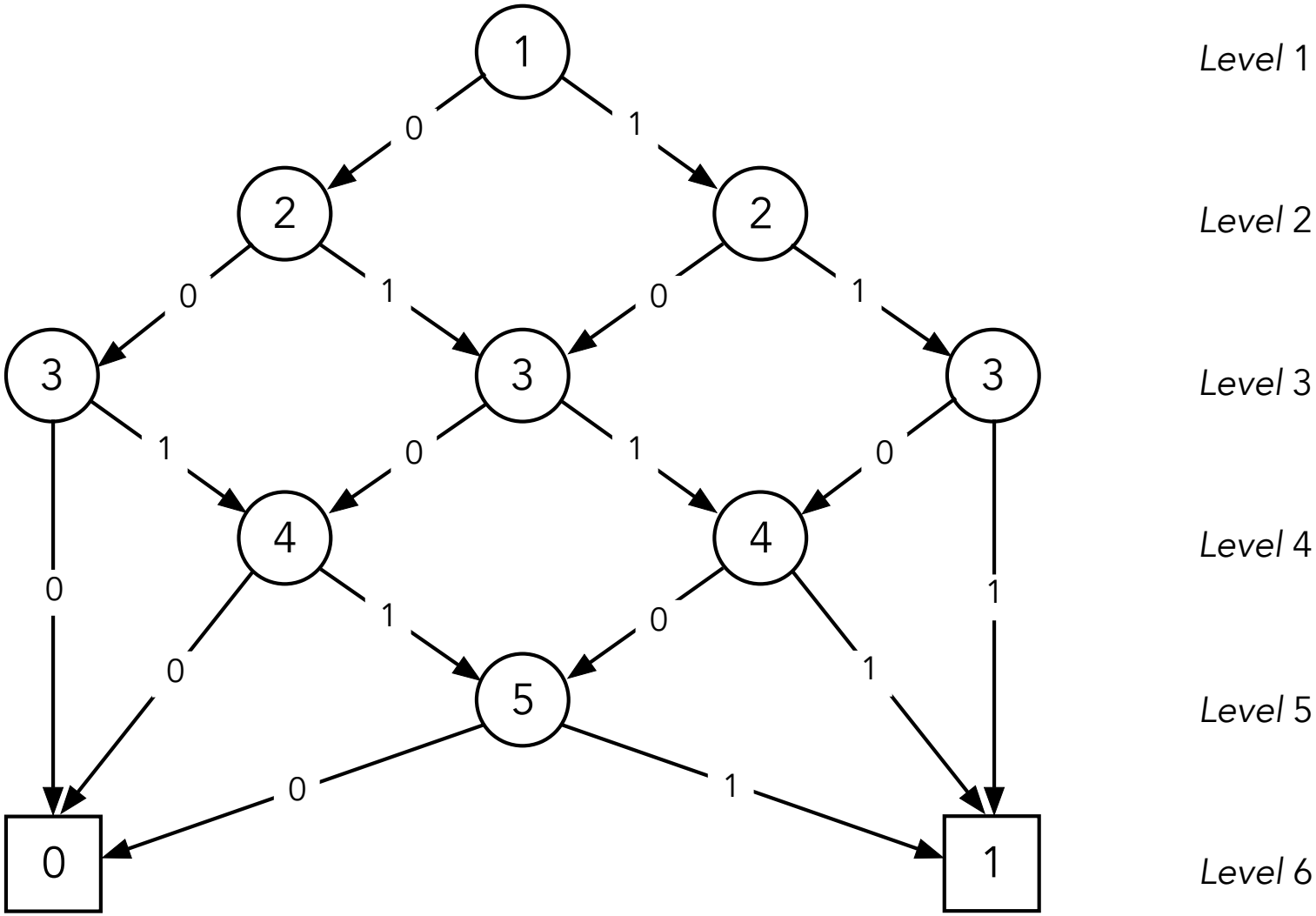


Figure: ROBDD of a 3-out-5-system.



Exercise 4.14

Explain briefly why the number of nodes in a reduced ordered binary decision diagram of a k-out-n-system grows polynomially in n.

SOLUTION: We let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be the vector of component state variables, and introduce the partial sums:

$$S_k = \sum_{i=1}^{k} X_i, \quad k = 1, \ldots, n.$$

If we do not merge trivial nodes with the leaf nodes, the number of nodes at level k in the diagram is determined by the number of different binary functions that can be formed at this level.

Furthermore, the number of different binary functions that can be formed at level k is determined by the number of states the partial sum S_{k-1} can be in.



Exercise 4.14 (cont.)

Since $S_{k-1} \in \{0, 1, \ldots, (k-1)\}$, the number of possible states S_{k-1} can be in is k.

Hence, the total number of intermediate nodes is bounded above by:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

From this we conclude that the number of nodes in a reduced ordered binary decision diagram of a k-out-n-system grows polynomially in n.

Example. Assume that $n = 25$. Then we have:

$$n(n+1)/2 = 25 \cdot 26/2 = 325$$

$$2^n = 2^{25} = 33\,554\,432$$

Thus, an ROBDD is much smaller than a BDD in this case!!



Exercise 4.14 (cont.)

```
from c_robdd import ROBDDSystem
from c_threshold import ROBDDThreshold

import matplotlib.pyplot as plt
import numpy as np

save_plots = True

# Component reliabilities
p = 0.5

k = 13
n = 25

# Number of plot points
points = 100

# Component weights
a = []
for _ in range(n):
    a.append(1)

sys = ROBDDSystem(ROBDDThreshold(a, n, k))

result = sys.calculateReliability0(p)
sys.printSystem()

print("")
print("System unreliability = ", result[0])
print("System reliability = ", result[1])
```



Exercise 4.14 (cont.)

```
sys = ROBDDSystem(ROBDDThreshold(a, n, k))

result = sys.calculateReliability0(p)
sys.printSystem()

print("")
print("System unreliability = ", result[0])
print("System reliability = ", result[1])

pp = np.linspace(0.0, 1.0, points)
uu = np.zeros(points)
hh = np.zeros(points)

for i in range(points):
    result = sys.calculateReliability0(pp[i])
    uu[i] = result[0]
    hh[i] = result[1]

fig = plt.figure(figsize = (7, 4))
plt.plot(pp, uu, label='u(p)')
plt.plot(pp, hh, label='h(p)')
plt.xlabel('p')
# plt.ylabel('h and u')
plt.title("Reliability / unreliability vs. p")
plt.legend()
if save_plots:
    plt.savefig("k_out_of_n/relplot_" + str(k) + "_" + str(n) + ".pdf")
plt.show()
```



Exercise 4.14 (cont.)

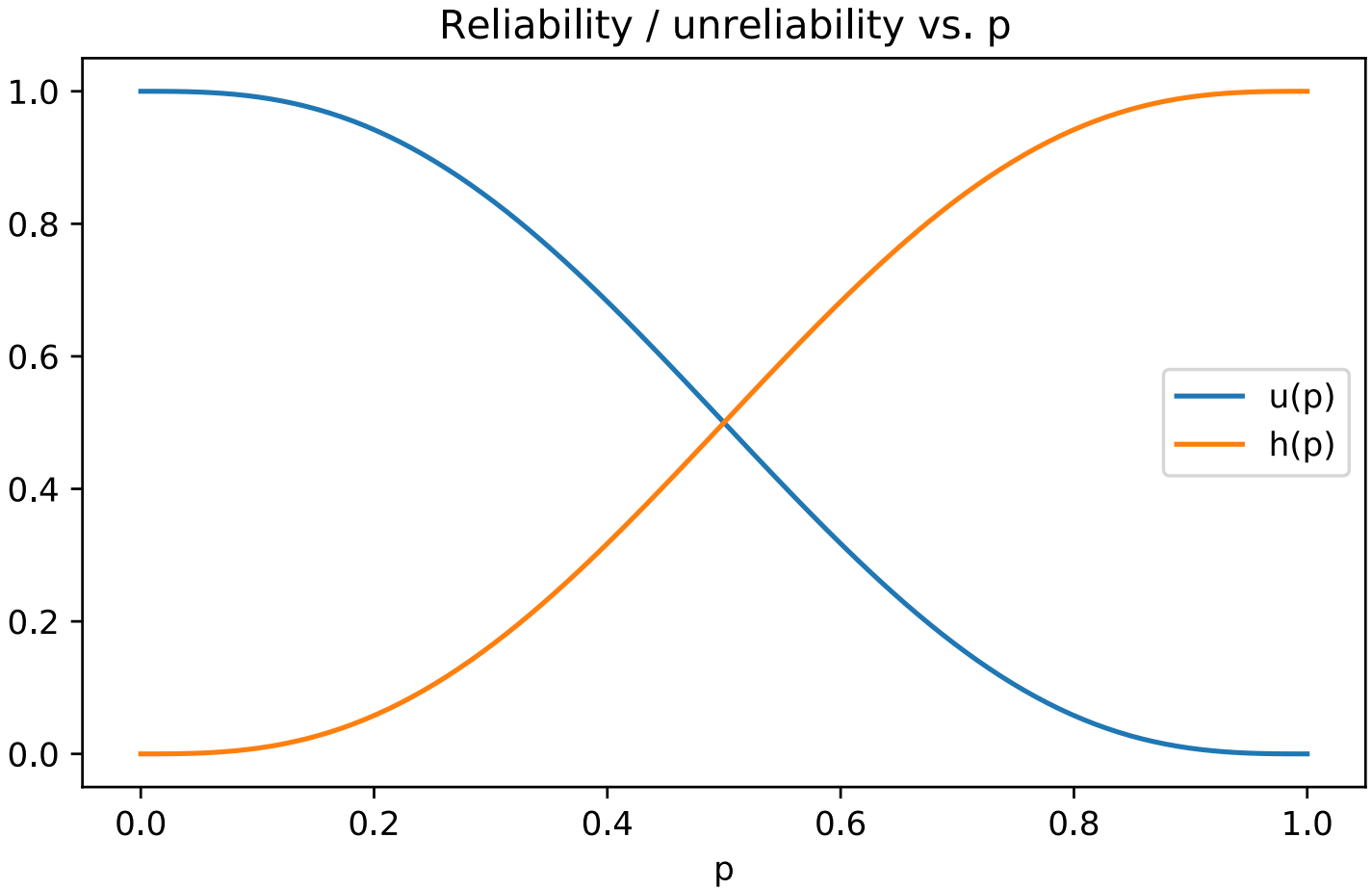


Figure: Reliability plot of a 3-out-5-system.



Exercise 4.14 (cont.)

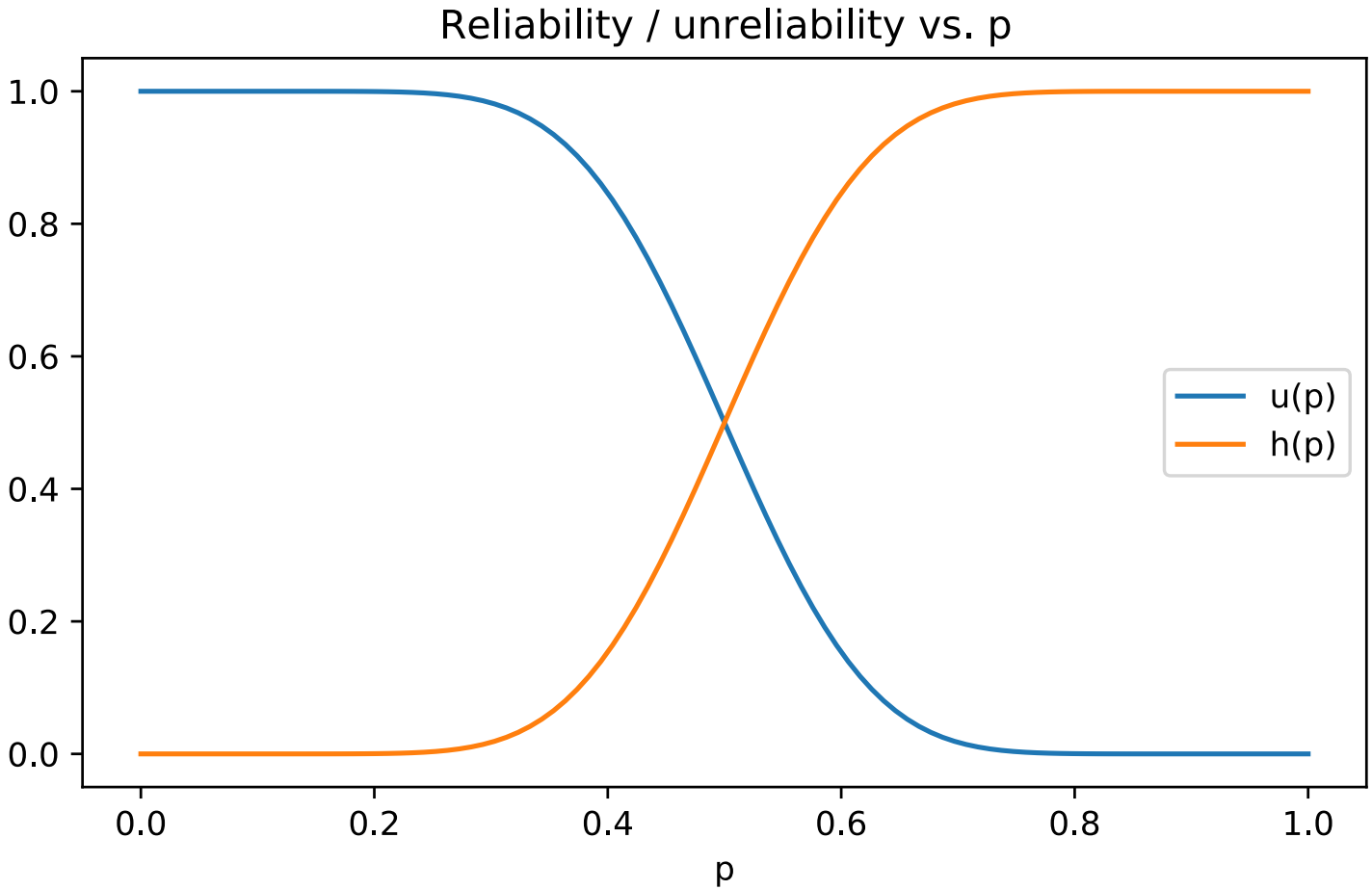


Figure: Reliability plot of a 13-out-25-system.



Exercise 4.15

Let (C, ϕ) be a threshold system with weights $a_1 = 8$, $a_2 = 6$, $a_3 = 6$, $a_4 = 4$, $a_5 = 4$, $a_6 = 2$ and threshold $b = 15$.

a) Assume that all components have reliability $p = 0.5$. Write a python script for calculating the reliability of the system. [*Answer*: $P(\phi = 1) = 0.5$.]



Exercise 4.15 (cont.)

```
from c_robdd import ROBDDSystem
from c_threshold import ROBDDThreshold

# Component reliabilities
p = 0.5

# Component weights
a = [8, 6, 6, 4, 4, 2]

# Threshold
b = 15

# Number of components
n = len(a)

# Sum of weights
wsum = 0
for wg in a:
    wsum += wg

sys = ROBDDSystem(ROBDDThreshold(a, wsum, b))

result = sys.calculateReliability0(p)
sys.printSystem()

print("")
print("System unreliability = ", result[0])
print("System reliability = ", result[1])
```



Exercise 4.15 (cont.)

```
C0[N0(1)]
C1[N0(0.5), N1(0.5)]
C2[N0(0.25), N1(0.25), N2(0.25), N3(0.25)]
C3[N0(0.25), N1(0.125), N2(0.125), N3(0.25)]
C4[N0(0.125), N1(0.125), N2(0.125)]
C5[N0(0.125)]

System unreliability =  0.5
System reliability =  0.5
```

Number of intermediate nodes = 15



Exercise 4.15 (cont.)

b) Run the script from (a) again, but with the weights in reversed order, i.e., $a_1 = 2$, $a_2 = 4$, $a_3 = 4$, $a_4 = 6$, $a_5 = 6$, $a_6 = 8$. Compare the number of intermediate nodes in the ROBDD with the number of nodes in ROBDD from (a).



Exercise 4.15 (cont.)

```
from c_robdd import ROBDDSystem
from c_threshold import ROBDDThreshold

# Component reliabilities
p = 0.5

# Component weights
a = [2, 4, 4, 6, 6, 8]

# Threshold
b = 15

# Number of components
n = len(a)

# Sum of weights
wsum = 0
for wg in a:
    wsum += wg

sys = ROBDDSystem(ROBDDThreshold(a, wsum, b))

result = sys.calculateReliability0(p)
sys.printSystem()

print("")
print("System unreliability = ", result[0])
print("System reliability = ", result[1])
```



Exercise 4.15 (cont.)

```
C0[N0(1)]
C1[N0(0.5), N1(0.5)]
C2[N0(0.25), N1(0.25), N2(0.25), N3(0.25)]
C3[N0(0.125), N1(0.25), N2(0.125), N3(0.125), N4(0.25), N5(0.125)]
C4[N0(0.1875), N1(0.125), N2(0.1875), N3(0.125), N4(0.0625), N5(0.0625), N6(0.125)]
C5[N0(0.15625), N1(0.15625), N2(0.09375), N3(0.09375)]

System unreliability =  0.5
System reliability =  0.5
```

Number of intermediate nodes = 24



Exercise 4.15 (cont.)

c) Assume instead that $p_1 = 0.82$, $p_2 = 0.78$, $p_3 = 0.66$, $p_4 = 0.91$, $p_5 = 0.73$, $p_6 = 0.88$. Modify the script from (a) and calculate the reliability of the system with these component reliabilities. [*Answer*: $P(\phi = 1) = 0.932025$.]



Exercise 4.15 (cont.)

```
from c_robdd import ROBDDSystem
from c_threshold import ROBDDThreshold

# Component reliabilities
p = [0.82, 0.78, 0.66, 0.91, 0.73, 0.88]

# Component weights
a = [8, 6, 6, 4, 4, 2]

# Threshold
b = 15

# Number of components
n = len(a)

# Sum of weights
wsum = 0
for wg in a:
    wsum += wg

sys = ROBDDSystem(ROBDDThreshold(a, wsum, b))

result = sys.calculateReliability(p)
sys.printSystem()

print("")
print("System unreliability = ", result[0])
print("System reliability = ", result[1])
```



Exercise 4.15 (cont.)

```
C0[N0(1)]
C1[N0(0.1800000000000005), N1(0.82)]
C2[N0(0.0396), N1(0.14040000000000005), N2(0.18039999999999998), N3(0.6396)]
C3[N0(0.07387200000000002), N1(0.09266400000000004), N2(0.06133599999999999), N3(0.33652799999999994)]
C4[N0(0.06722352000000002), N1(0.06415552), N2(0.030287519999999984)]
C5[N0(0.0572508000000001)]

System unreliability =  0.0679751568
System reliability =  0.9320248432000001
```



Exercise 5.1

Let (C, ϕ) be a k-out-of-n system, where $C = \{1, \ldots, n\}$. Prove that all the components of this system have the same Birnbaum measure for structural importance.

SOLUTION: Let $i \in C$ be a component in the system. We then recall that:

$$\phi(\boldsymbol{x}) = \mathrm{I}(\sum_{j \in C} x_j \geq k)$$

From this it follows that:

$$\phi(1_i, \boldsymbol{x}) = \mathrm{I}(1 + \sum_{j \in C \setminus i} x_j \geq k) = \mathrm{I}(\sum_{j \in C \setminus i} x_j \geq k - 1)$$

$$\phi(0_i, \boldsymbol{x}) = \mathrm{I}(\sum_{j \in C \setminus i} x_j \geq k)$$

Hence, we have:

$$\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x}) = \mathrm{I}(\sum_{j \in C \setminus i} x_j = k - 1)$$



Exercise 5.1 (cont.)

We then insert this into the formula for $J_B^{(i)}$ and get:

$$J_B^{(i)} = \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} (\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x}))$$

$$= \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} \mathrm{I}(\sum_{j \in C \setminus i} x_j = k - 1)$$

The number of non-zero terms in the sum is equal to the number of ways to choose $k-1$ components out of the $n-1$ components in $C \setminus i$.

Hence, we get:

$$J_B^{(i)} = \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} \mathrm{I}(\sum_{j \in C \setminus i} x_j = k - 1) = \frac{1}{2^{n-1}} \binom{n-1}{k-1}$$

Since this holds for all $i \in C$, we conclude that all components have the same structural importance.



Exercise 5.2

Compute $J_B^{(i)}$ for the components of the bridge structure and compare their structural importance.

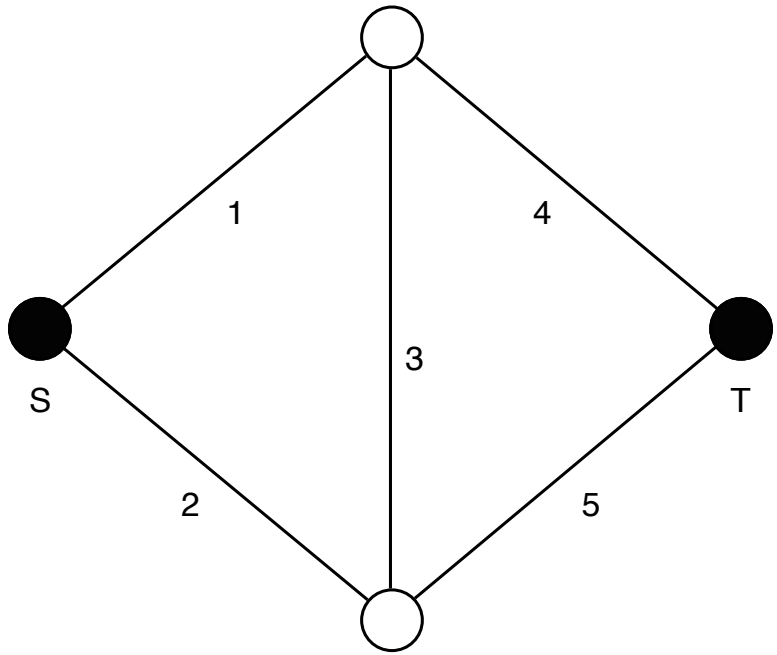
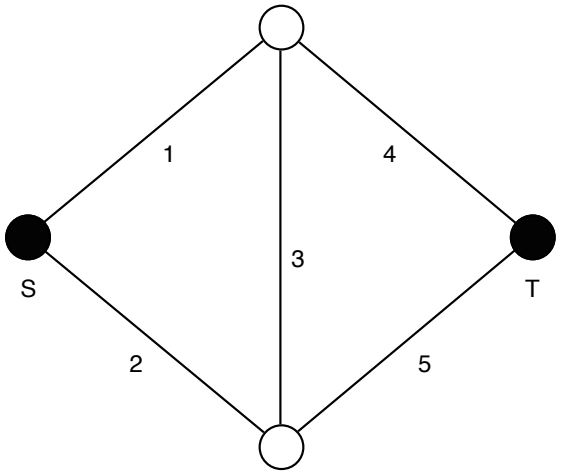


Figure: A bridge structure



Exercise 5.2 (cont.)



There are 2 critical path vectors for Component 3:

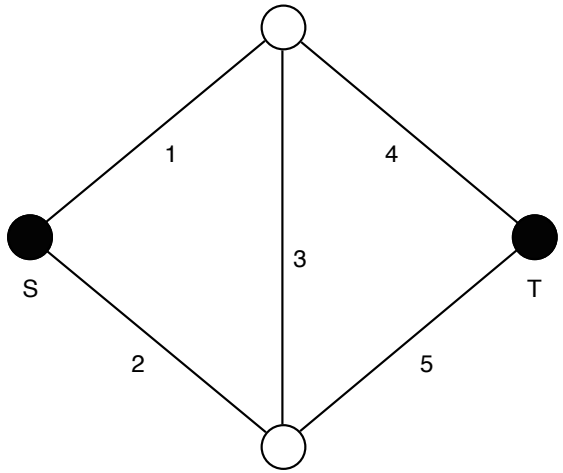
$$(1, 0, \cdot, 0, 1), \quad (0, 1, \cdot, 1, 0)$$

Hence, we get:

$$J_B^{(3)} = \frac{2}{2^{5-1}} = \frac{2}{16} = \frac{1}{8}$$



Exercise 5.2 (cont.)



There are 6 critical path vectors for Component 1:

$$(\cdot, 0, 0, 1, 0), \quad (\cdot, 1, 0, 1, 0), \quad (\cdot, 0, 1, 1, 0)$$
$$(\cdot, 0, 0, 1, 1), \quad (\cdot, 0, 1, 0, 1), \quad (\cdot, 0, 1, 1, 1)$$

Hence, we get:

$$J_B^{(1)} = \frac{6}{2^{5-1}} = \frac{6}{16} = \frac{3}{8}$$

By symmetry the remaining components have the same structural importance as Component 1.



Exercise 5.3

Consider a binary monotone system(C, ϕ), where $C = \{1, \ldots, n\}$. Assume that the *i*'th component be in *series* with the rest of the system, while the *j*'th component is *not*.

Prove that:

$$J_B^{(i)} > J_B^{(j)}.$$



Exercise 5.3 (cont.)

Since i is in series with the rest of the system, we have:

$$\phi(0_i, \boldsymbol{x}) = 0, \quad \text{for all } (\cdot_i, \boldsymbol{x}) \in \{0,1\}^{n-1}$$

Since j is *not* in series with the rest of the system, we have:

$$\phi(0_j, \boldsymbol{x}) = 1, \quad \text{for at least one } (\cdot_j, \boldsymbol{x}) \in \{0,1\}^{n-1}$$

Hence, we then get:

$$\begin{aligned} 2^{n-1} J_B^{(i)} &= \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})] = \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) + \phi(0_i, \boldsymbol{x})] \\ &= \sum_{\boldsymbol{x}} \phi(\boldsymbol{x}) = \sum_{(\cdot_j, \boldsymbol{x})} [\phi(1_j, \boldsymbol{x}) + \phi(0_j, \boldsymbol{x})] \\ &> \sum_{(\cdot_j, \boldsymbol{x})} [\phi(1_j, \boldsymbol{x}) - \phi(0_j, \boldsymbol{x})] = 2^{n-1} J_B^{(j)} \end{aligned}$$



Exercise 5.4

Compute $I_B^{(i)}$ for the components of the bridge structure and compare their reliability importance.

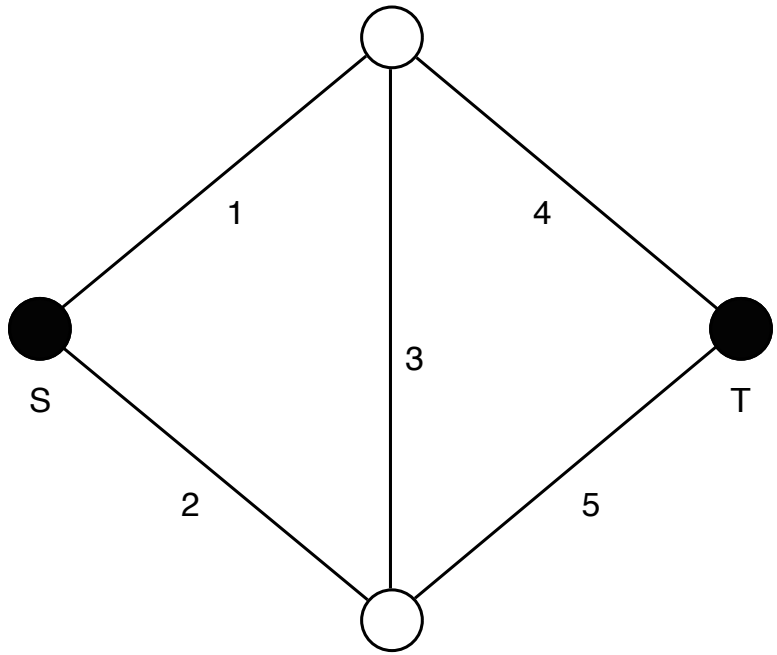


Figure: A bridge structure



Exercise 5.4 (cont.)

SOLUTION: We know that:

$$\begin{aligned} h(\boldsymbol{p}) &= p_3(p_1 \amalg p_2)(p_4 \amalg p_5) + (1-p_3)((p_1p_4) \amalg (p_2p_5)) \\ &= p_3(p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5) \\ &+ (1-p_3)(p_1p_4 + p_2p_5 - p_1p_2p_4p_5). \end{aligned}$$

Hence, we get:

$$I_B^{(1)} = \frac{\partial h(\boldsymbol{p})}{\partial p_1} = p_3(1-p_2)(p_4 + p_5 - p_4p_5) + (1-p_3)(p_4 - p_2p_4p_5)$$

$$I_B^{(2)} = \frac{\partial h(\boldsymbol{p})}{\partial p_2} = p_3(1-p_1)(p_4 + p_5 - p_4p_5) + (1-p_3)(p_5 - p_1p_4p_5)$$



Exercise 5.4 (cont.)

$$I_B^{(3)} = \frac{\partial h(\boldsymbol{p})}{\partial p_3} = (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5) - (p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5)$$

$$I_B^{(4)} = \frac{\partial h(\boldsymbol{p})}{\partial p_4} = p_3(p_1 + p_2 - p_1 p_2)(1 - p_5) + (1 - p_3)(p_1 - p_1 p_2 p_5)$$

$$I_B^{(5)} = \frac{\partial h(\boldsymbol{p})}{\partial p_5} = p_3(p_1 + p_2 - p_1 p_2)(1 - p_4) + (1 - p_3)(p_2 - p_1 p_2 p_4)$$

Assuming that all components have the same reliability, that is, $p_i = p$, $i = 1, \ldots, 5$, we get:

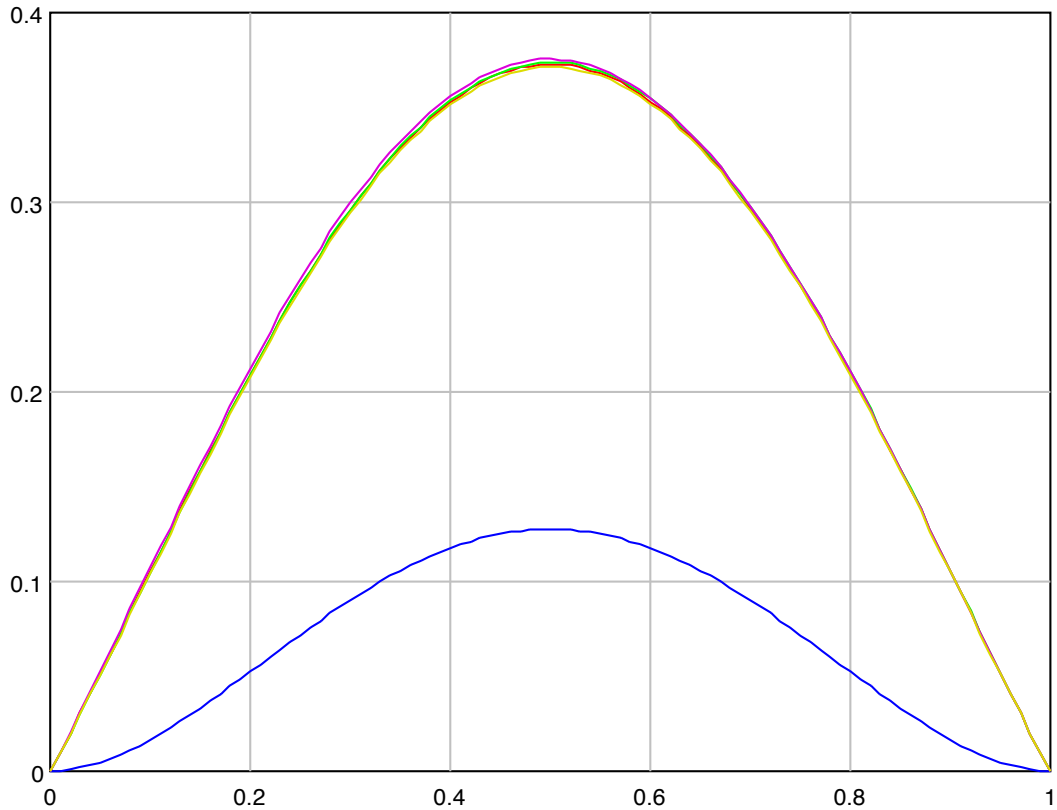
$$\begin{aligned} I_B^{(i)} &= p(2p - p^2)(1 - p) + (1 - p)(p - p^3) \\ &= p(1 - p)(1 + 2p - 2p^2), \quad i = 1, 2, 4, 5, \end{aligned}$$

$$I_B^{(3)} = (2p - p^2)^2 - (2p^2 - p^4)$$



Exercise 5.4 (cont.)

In order to compare the reliability importance of the components, we plot $I_B^{(i)}$ as functions of the common component reliability p:



The blue curve represents $I_B^{(3)}$, while the other curves represent

Exercise 5.4 (cont.)

Assume instead that $p_1 = p_3 = p_5 = 0.9$, while $p_2 = p_4 = 0.1$. In this case we get that:

$$
\begin{aligned}
I_B^{(1)} &= p_3(1-p_2)(p_4+p_5-p_4p_5) + (1-p_3)(p_4 - p_2p_4p_5) \\
&= 0.9(1-0.1)(0.1+0.9-0.1\cdot 0.9) + (1-0.9)(0.1 - 0.1\cdot 0.1\cdot 0.9) \\
&= 0.9^2(1.0-0.09) + 0.1(0.1-0.009) = 0.7371 + 0.0091 = 0.7462
\end{aligned}
$$

$$
\begin{aligned}
I_B^{(2)} &= p_3(1-p_1)(p_4+p_5-p_4p_5) + (1-p_3)(p_5 - p_1p_4p_5) \\
&= 0.9(1-0.9)(0.1+0.9-0.1\cdot 0.9) + (1-0.9)(0.9 - 0.9\cdot 0.1\cdot 0.9) \\
&= 0.9\cdot 0.1\cdot(1.0-0.09) + 0.1\cdot(0.9-0.081) = 0.0819 + 0.0819 = 0.1638
\end{aligned}
$$

$$
\begin{aligned}
I_B^{(3)} &= (p_1+p_2-p_1p_2)(p_4+p_5-p_4p_5) - (p_1p_4 + p_2p_5 - p_1p_2p_4p_5) \\
&= (0.9+0.1-0.9\cdot 0.1)^2 - (0.9\cdot 0.1 + 0.1\cdot 0.9 - 0.9^2\cdot 0.1^2) \\
&= (1.0-0.09)^2 - (0.18 - 0.09^2) = 0.8281 - 0.1719 = 0.6562
\end{aligned}
$$



Exercise 5.4 (cont.)

By symmetry we also get that:

$$I_B^{(4)} = I_B^{(2)} = 0.1638$$

$$I_B^{(5)} = I_B^{(1)} = 0.7462$$

Hence, with these component reliabilities we have the following ranking of importance:

$$I_B^{(1)} = I_B^{(5)} > I_B^{(3)} > I_B^{(2)} = I_B^{(4)}$$

