STK3405 – Lecture 2

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Section 2.3

Dual systems



Dual systems

Definition

Let ϕ be a structure function of a binary monotone system of order n. We then define the dual structure function, ϕ^D for all $\boldsymbol{x} \in \{0,1\}^n$ as:

$$\phi^D(\boldsymbol{x}) = 1 - \phi(\mathbf{1} - \boldsymbol{x}).$$

Furthermore, if $\boldsymbol{x}$ is a component state vector, we define the dual component state vector $\boldsymbol{x}^D$ as:

$$\boldsymbol{x}^D = (x_1^D, \ldots, x_n^D) = (1 - x_1, \ldots, 1 - x_n) = \mathbf{1} - \boldsymbol{x}$$



Dual systems (cont.)

Note:

- The relation between ϕ and ϕ^D is a relation between two functions
- The relation between $\boldsymbol{x}$ and $\boldsymbol{x}^D$ is a relation between two component state vectors

We have the following relation between $\phi(\boldsymbol{x})$ and $\phi^D(\boldsymbol{x}^D)$:

$$\phi^D(\boldsymbol{x}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{x}^D) = 1 - \phi(\boldsymbol{x}).$$

Hence, the dual system is functioning if and only if the original system is failed and vice versa.



Examples of dual systems

Let ϕ be the structure function of a system of order 3 such that:

$$\phi(\boldsymbol{x}) = x_1 \amalg (x_2 \cdot x_3),$$

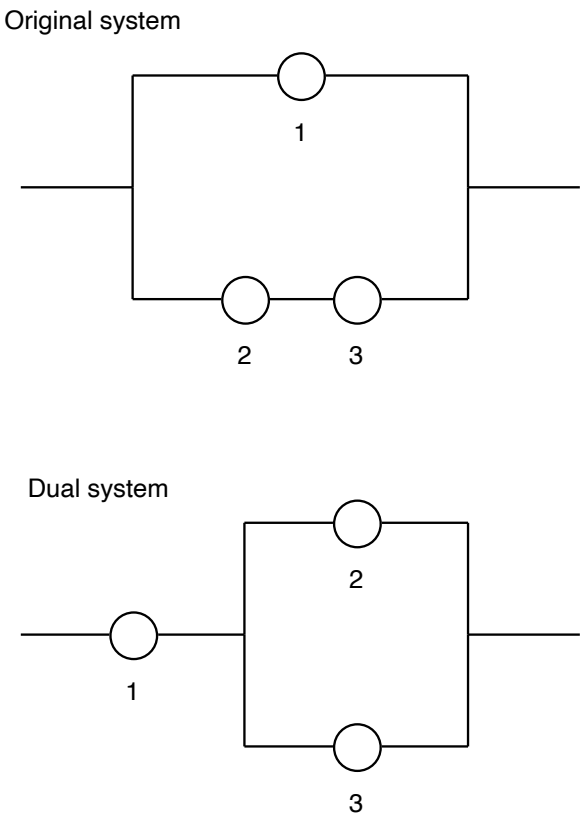
The dual structure function is then given by:

$$\begin{aligned}
\phi^D(\boldsymbol{x}) &= 1 - \phi(\boldsymbol{1} - \boldsymbol{x}) \\
&= 1 - (1 - x_1) \amalg ((1 - x_2) \cdot (1 - x_3)) \\
&= 1 - [1 - (1 - (1 - x_1))(1 - (1 - x_2) \cdot (1 - x_3))] \\
&= 1 - [1 - x_1 \cdot [1 - (1 - x_2) \cdot (1 - x_3)]] \\
&= x_1 \cdot [1 - (1 - x_2) \cdot (1 - x_3)] \\
&= x_1 \cdot (x_2 \amalg x_3)
\end{aligned}$$



Examples of dual systems (cont.)

$$\phi(\boldsymbol{x}) = x_1 \amalg (x_2 \cdot x_3), \qquad \phi^D(\boldsymbol{x}) = x_1 \cdot (x_2 \amalg x_3)$$



Examples of dual systems (cont.)

Let (C, ϕ) be a series system of order n:

$$\phi(\boldsymbol{x}) = \prod_{i=1}^{n} x_i.$$

The dual structure function is then given by:

$$\begin{aligned}\phi^D(\boldsymbol{x}) &= 1 - \phi(\mathbf{1} - \boldsymbol{x}) \\ &= 1 - \prod_{i=1}^{n}(1 - x_i) = \coprod_{i=1}^{n} x_i.\end{aligned}$$

Thus, the dual system is a parallel system of order n.



Examples of dual systems (cont.)

Let (C, ϕ) be a parallel system of order n:

$$\phi(\boldsymbol{x}) = \coprod_{i=1}^{n} x_i.$$

The dual structure function is then given by:

$$\begin{aligned}\phi^D(\boldsymbol{x}) &= 1 - \phi(\mathbf{1} - \boldsymbol{x}) = 1 - \coprod_{i=1}^{n}(1 - x_i) \\ &= 1 - (1 - \prod_{i=1}^{n}(1 - (1 - x_i)) = \prod_{i=1}^{n} x_i.\end{aligned}$$

Thus, the dual system is a series system of order n.



Examples of dual systems (cont.)

Let (C, ϕ) be a 2-out-of-3 system. Then the structure function is given by:

$$\begin{aligned}
\phi(\boldsymbol{x}) &= (x_1 x_2) \amalg (x_1 x_3) \amalg (x_2 x_3) \\
&= 1 - (1 - x_1 x_2)(1 - x_1 x_3)(1 - x_2 x_3) \\
&= 1 - [1 - x_1 x_2 - x_1 x_3 - x_2 x_3 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 - x_1^2 x_2^2 x_3^2] \\
&= 1 - [1 - x_1 x_2 - x_1 x_3 - x_2 x_3 + 2 x_1 x_2 x_3] \\
&= x_1 x_2 + x_1 x_3 + x_2 x_3 - 2 x_1 x_2 x_3
\end{aligned}$$

The dual structure function is then given by:

$$\phi^D(\boldsymbol{x}) = 1 - \phi(\mathbf{1} - \boldsymbol{x})$$



Examples of dual systems (cont.)

$$= 1 - [(1 - x_1)(1 - x_2) + (1 - x_1)(1 - x_3) + (1 - x_2)(1 - x_3) - 2(1 - x_1)(1 - x_2)(1 - x_3)]$$

$$= 1 - [(1 - x_1 - x_2 + x_1 x_2) + (1 - x_1 - x_3 + x_1 x_3) + (1 - x_2 - x_3 + x_2 x_3) - 2(1 - x_1 - x_2 - x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 - x_1 x_2 x_3)]$$

$$= 1 - [3 - 2x_1 - 2x_2 - 2x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 - 2 + 2x_1 + 2x_2 + 2x_3 - 2x_1 x_2 - 2x_1 x_3 - 2x_2 x_3 + 2x_1 x_2 x_3]$$

$$= 1 - [1 - x_1 x_2 - x_1 x_3 - x_2 x_3 + 2x_1 x_2 x_3]$$

$$= x_1 x_2 + x_1 x_3 + x_2 x_3 - 2x_1 x_2 x_3 = \phi(\boldsymbol{x})$$

Thus, the dual system is also a 2-out-of-3 system.



Dual systems (cont.)

Theorem

Let ϕ be the structure function of a binary monotone system, and let ϕ^D be the corresponding dual structure function. Then we have:

$$(\phi^D)^D = \phi.$$

That is, the dual of the dual system is equal to the original system.

Proof: For all $\boldsymbol{x} \in \{0,1\}^n$ we have:

$$\begin{aligned}(\phi^D)^D(\boldsymbol{x}) &= 1 - \phi^D(\mathbf{1} - \boldsymbol{x}) \\ &= 1 - [1 - \phi(\mathbf{1} - (\mathbf{1} - \boldsymbol{x}))] \\ &= \phi(\boldsymbol{x}).\end{aligned}$$



Section 2.4

Reliability of binary monotone systems



Reliability of binary monotone systems

Let (C, ϕ) be a binary monotone system, and introduce *stochastic component state variables*, $X_1, \ldots, X_n$. We then let:

$$p_i = P(X_i = 1) = \text{The } \textit{reliability} \text{ of a component } i, \quad i \in C.$$

Since the state variable X_i is binary, we have for all $i \in C$:

$$\mathrm{E}[X_i] = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1) = p_i$$

Thus, the reliability of component i is equal to the expected value of its component state variable, X_i.



Reliability of binary monotone systems (cont.)

$$h = P(\phi(\boldsymbol{X}) = 1) = \text{The reliability of the system}$$

Since ϕ is binary, we have:

$$\mathrm{E}[\phi(\boldsymbol{X})] = 0 \cdot P(\phi(\boldsymbol{X}) = 0) + 1 \cdot P(\phi(\boldsymbol{X}) = 1) = P(\phi(\boldsymbol{X}) = 1) = h.$$

Thus, the reliability of the system is equal to the expected value of the structure function, $\phi(\boldsymbol{X})$.

From this it immediately follows that the reliability of a system, at least in principle, can be calculated as:

$$h = \mathrm{E}[\phi(\boldsymbol{X})] = \sum_{\boldsymbol{X} \in \{0,1\}^n} \phi(\boldsymbol{x}) P(\boldsymbol{X} = \boldsymbol{x})$$



Independent components

We now focus on the case where the component state variables can be assumed to be independent and introduce $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$. We note that:

$$P(X_i = x_i) = \begin{cases} p_i & \text{if } x_i = 1, \\ 1 - p_i & \text{if } x_i = 0. \end{cases}$$

Since x_i is either 0 or 1, $P(X_i = x_i)$ can be written in the following more compact form:

$$P(X_i = x_i) = p_i^{x_i}(1 - p_i)^{1-x_i}.$$



The reliability function

Thus, when the component state variables are independent, their joint distribution can be written as:

$$P(\boldsymbol{X} = \boldsymbol{x}) = \prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}.$$

Hence, we get the following expression for the system reliability:

$$h = h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \sum_{\boldsymbol{x} \in \{0,1\}^n} \phi(\boldsymbol{x}) \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}$$

The function $h(\boldsymbol{p})$ is called the reliability function of the system.



Reliability of a series system

Consider a series system of order n. Assuming that the component state variables are independent, the reliability of this system is given by:

$$h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \mathrm{E}[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n} \mathrm{E}[X_i] = \prod_{i=1}^{n} p_i,$$

where the third equality follows since $X_1, X_2, \ldots, X_n$ are independent.



Reliability of a parallel system

Consider a parallel system of order n. Assuming that the component state variables are independent, the reliability of this system is given by:

$$h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \mathrm{E}[\coprod_{i=1}^{n} X_i] = \mathrm{E}[1 - \prod_{i=1}^{n}(1 - X_i)]$$

$$= 1 - \prod_{i=1}^{n}(1 - E[X_i]) = \coprod_{i=1}^{n} \mathrm{E}[X_i] = \coprod_{i=1}^{n} p_i,$$

where the fourth equality follows since $X_1, X_2, \ldots, X_n$ are independent.

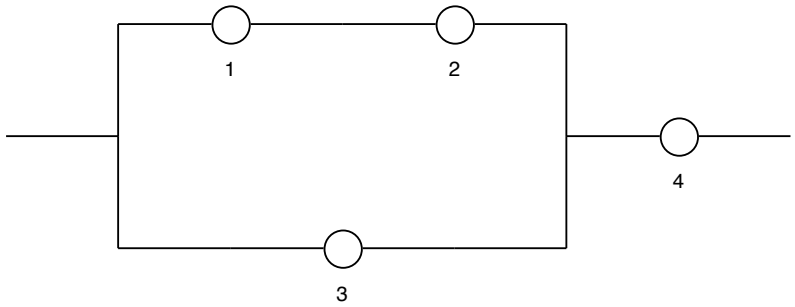
NOTE: This implies that the coproduct operator satisfies:

$$E[Z_1 \amalg \cdots \amalg Z_n] = E[Z_1] \amalg \cdots \amalg E[Z_n]$$

for independent variables $Z_1, \ldots, Z_n$ (just as the product operator).



Reliability of a mixed system



Assuming independent component states the system reliability becomes:

$$
\begin{aligned}
h(\boldsymbol{p}) &= \mathrm{E}[\phi(\boldsymbol{X})] = \mathrm{E}[[(X_1 \cdot X_2) \amalg X_3] \cdot X_4] \\
&= \mathrm{E}[(X_1 \cdot X_2) \amalg X_3] \cdot \mathrm{E}[X_4] \\
&= [\mathrm{E}[X_1 \cdot X_2] \amalg \mathrm{E}[X_3]] \cdot \mathrm{E}[X_4] \\
&= [(\mathrm{E}[X_1] \cdot \mathrm{E}[X_2]) \amalg \mathrm{E}[X_3]] \cdot \mathrm{E}[X_4] \\
&= [(p_1 \cdot p_2) \amalg p_3] \cdot p_4.
\end{aligned}
$$



Component level changes vs. system level changes

In the following we define $\boldsymbol{p} \cdot \boldsymbol{p}'$ as $(p_1 \cdot p_1', \ldots, p_n \cdot p_n')$.

Theorem

Let $h(\boldsymbol{p})$ be the reliability function of a binary monotone system (C, ϕ) of order n. Then for all $\boldsymbol{p}, \boldsymbol{p}' \in [0,1]^n$ we have:

(*i*) $h(\boldsymbol{p} \amalg \boldsymbol{p}') \geq h(\boldsymbol{p}) \amalg h(\boldsymbol{p}')$,

(*ii*) $h(\boldsymbol{p} \cdot \boldsymbol{p}') \leq h(\boldsymbol{p}) \cdot h(\boldsymbol{p}')$

If (C, ϕ) is coherent, equality holds in (i) for all $\boldsymbol{p}, \boldsymbol{p}' \in [0,1]^n$ if and only if (C, ϕ) is a parallel system.

If (C, ϕ) is coherent, equality holds in (ii) for all $\boldsymbol{p}, \boldsymbol{p}' \in [0,1]^n$ if and only if (C, ϕ) is a series system.

Component level vs. system level (cont.)

Proof: Assume that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are two independent component state vectors with corresponding reliability vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$ respectively.

Since $E[Z_1 \amalg Z_2] = E[Z_1] \amalg E[Z_2]$ when Z_1 and Z_2 are independent, we get that:

$$\begin{aligned} h(\boldsymbol{p} \amalg \boldsymbol{p}') - h(\boldsymbol{p}) \amalg h(\boldsymbol{p}') \\ &= E[\phi(\boldsymbol{X} \amalg \boldsymbol{Y})] - E[\phi(\boldsymbol{X})] \amalg E[\phi(\boldsymbol{Y})] \\ &= E[\phi(\boldsymbol{X} \amalg \boldsymbol{Y}) - \phi(\boldsymbol{X}) \amalg \phi(\boldsymbol{Y})], \end{aligned}$$

where the last expectation must be non-negative since by the corresponding result for structure functions we know that:

$$\phi(\boldsymbol{x} \amalg \boldsymbol{y}) - \phi(\boldsymbol{x}) \amalg \phi(\boldsymbol{y}) \geq 0, \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^n.$$

This completes the proof of (i). The proof of (ii) is similar.



Component level vs. system level (cont.)

We now consider the case where (C, ϕ) is coherent and show that equality in (i) holds for all $\boldsymbol{p}, \boldsymbol{p}' \in [0,1]^n$ if and only if (C, ϕ) is a parallel system.

Assume that $0 < p_i < 1, 0 < p_i' < 1$ for $i = 1, \ldots, n$. This implies that:

$$P(\boldsymbol{X} = \boldsymbol{x}, \boldsymbol{Y} = \boldsymbol{y}) > 0, \text{ for all } \boldsymbol{x} \in \{0,1\}^n \text{ and } \boldsymbol{y} \in \{0,1\}^n.$$

From this it follows that:

$$E[\phi(\boldsymbol{X} \amalg \boldsymbol{Y}) - \phi(\boldsymbol{X}) \amalg \phi(\boldsymbol{Y})] = 0$$

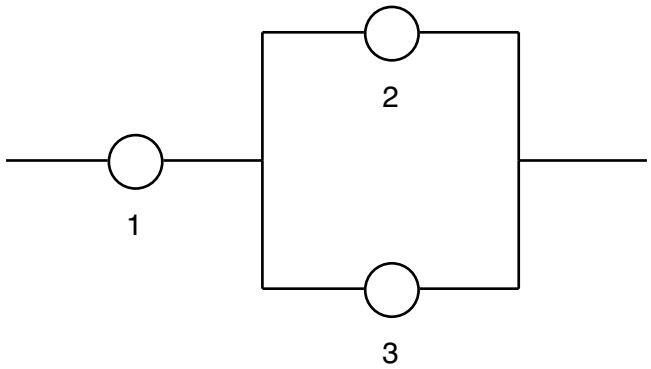
if and only if

$$\phi(\boldsymbol{x} \amalg \boldsymbol{y}) - \phi(\boldsymbol{x}) \amalg \phi(\boldsymbol{y}) = 0 \text{ for all } \boldsymbol{x} \in \{0,1\}^n \text{ and } \boldsymbol{y} \in \{0,1\}^n.$$

By the corresponding result for structure functions this holds if and only if (C, ϕ) is a parallel system. The other equivalence is proved similarly.



Component level vs. system level (cont.)



Let (C, ϕ) be a system with independent component state variables with $P(X_i = 1) = p$ for all $i \in C$, and where $\phi(\boldsymbol{x}) = x_1 \cdot (x_2 \amalg x_3)$.

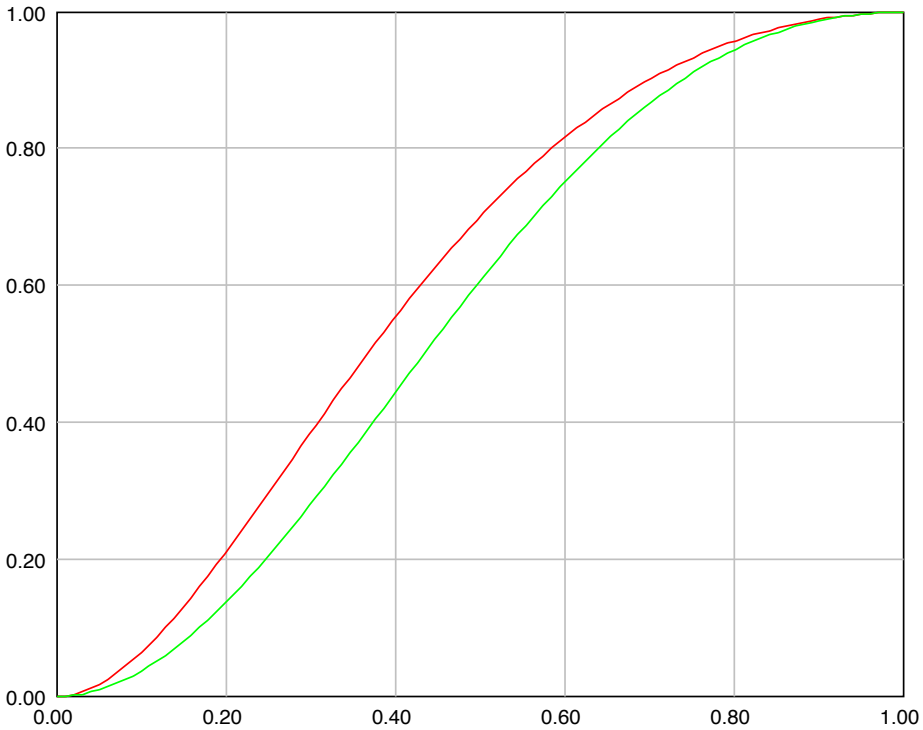
We then get that $h(\boldsymbol{p}) = h(p) = p \cdot (p \amalg p) = p \cdot (p + p - p^2) = 2p^2 - p^3$.

Hence, for all $0 \leq p \leq 1$, we have:

$$\begin{aligned} h(\boldsymbol{p} \amalg \boldsymbol{p}') &= 2(p \amalg p)^2 - (p \amalg p)^3 \\ &\geq h(\boldsymbol{p}) \amalg h(\boldsymbol{p}') = (2p^2 - p^3) \amalg (2p^2 - p^3) \end{aligned}$$



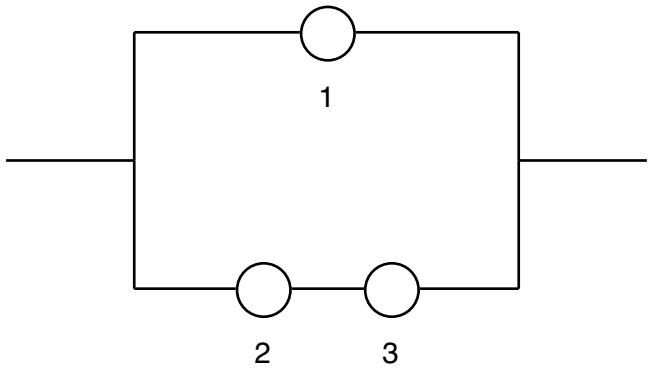
Component level vs. system level (cont.)



- Red curve: $h(\boldsymbol{p} \amalg \boldsymbol{p}') = 2(p \amalg p)^2 - (p \amalg p)^3$
- Green curve: $h(\boldsymbol{p}) \amalg h(\boldsymbol{p}') = (2p^2 - p^3) \amalg (2p^2 - p^3)$



Component level vs. system level (cont.)



Let (C, ϕ) be a system with independent component state variables with $P(X_i = 1) = p$ for all $i \in C$, and where $\phi(\boldsymbol{x}) = x_1 \amalg (x_2 \cdot x_3)$.

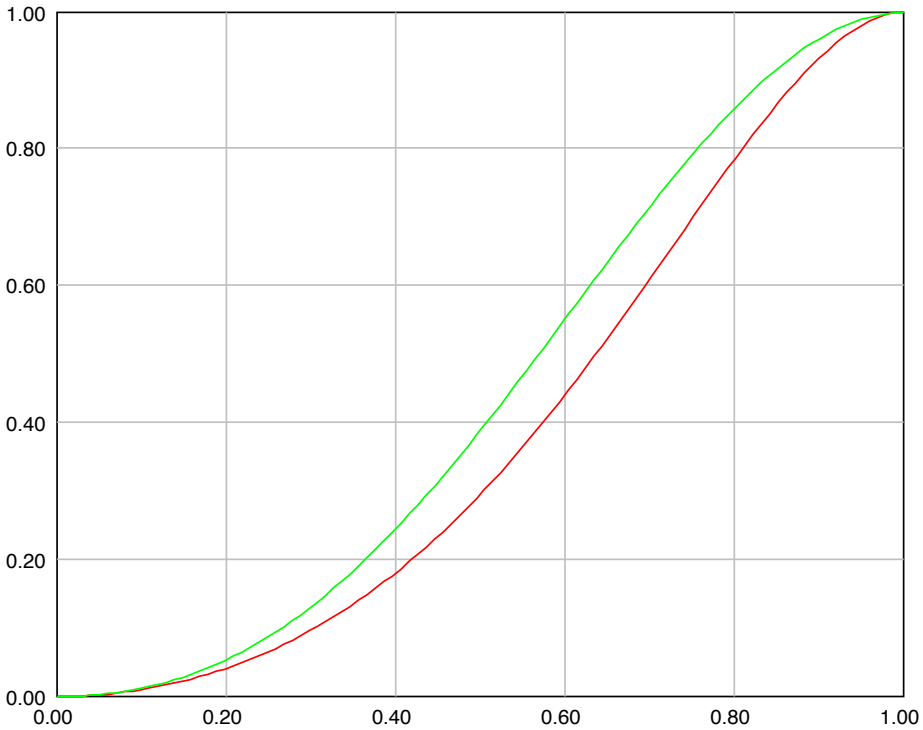
We then get that $h(\boldsymbol{p}) = h(p) = p \amalg (p \cdot p) = p \amalg p^2 = p + p^2 - p^3$.

Hence, for all $0 \leq p \leq 1$, we have:

$$\begin{aligned} h(\boldsymbol{p} \cdot \boldsymbol{p}') &= p^2 + p^4 - p^6 \\ &\leq h(\boldsymbol{p}) \cdot h(\boldsymbol{p}') = (p + p^2 - p^3)^2 \end{aligned}$$



Component level vs. system level (cont.)



- Red curve: $h(\boldsymbol{p} \cdot \boldsymbol{p}') = p^2 + p^4 - p^6$
- Green curve: $h(\boldsymbol{p}) \cdot h(\boldsymbol{p}') = (p + p^2 - p^3)^2$



Section 2.5

k-out-of-*n* systems



k-out-of-n systems

A k-out-of-n system is a binary monotone system (C, ϕ) where $C = \{1, \ldots, n\}$ which functions if and only if at least k out of the n components are functioning.

Let the component state variable of component i be x_i, $i \in C$, and let the vector of component state variables be $\boldsymbol{x} = (x_1, \ldots, x_n)$.

The structure function, ϕ, can then be written:

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{otherwise.} \end{cases}$$



An *n*-out-of-*n* system = A series system

An *n*-out-of-*n* system is the same as a series system:

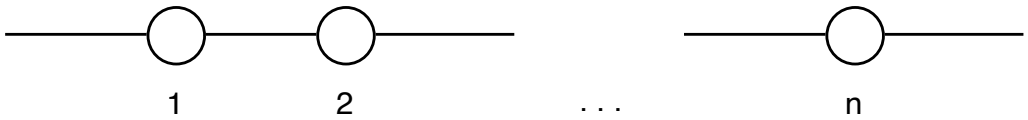


Figure: A reliability block diagram of an *n*-out-of-*n* system.



A 1-out-of-*n* system = A parallel system

A 1-out-of-*n* system is the same as a parallel system:

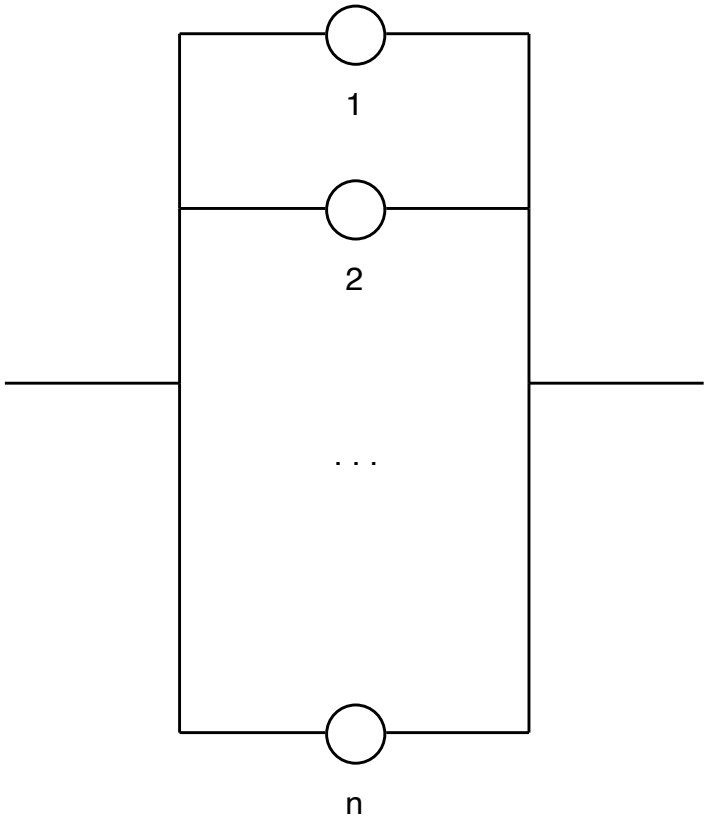


Figure: A reliability block diagram of an 1-out-of-*n* system.



A 2-out-of-3 system

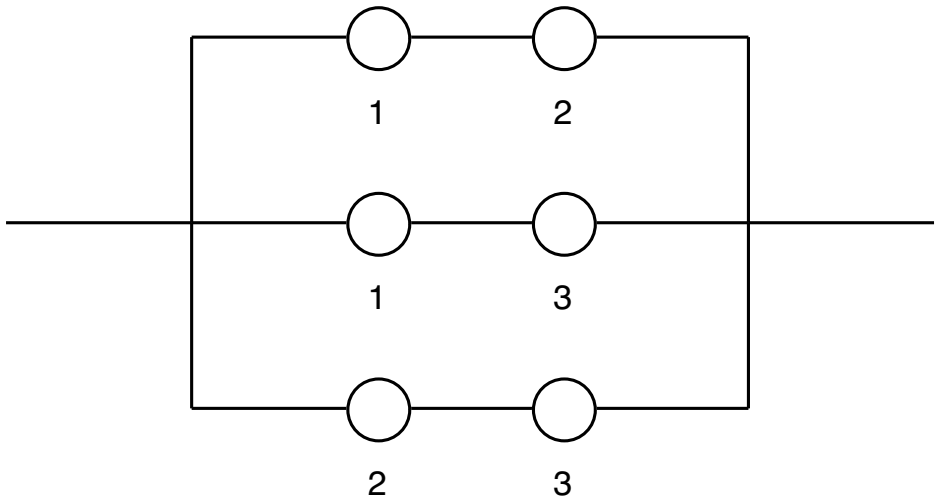


Figure: A reliability block diagram of a 2-out-of-3 system.

For a 2-out-of-3 system to function 2 out of 3 components must function. There are 3 possible subsets of components which contains 2 components: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$.



A 2-out-of-3 system

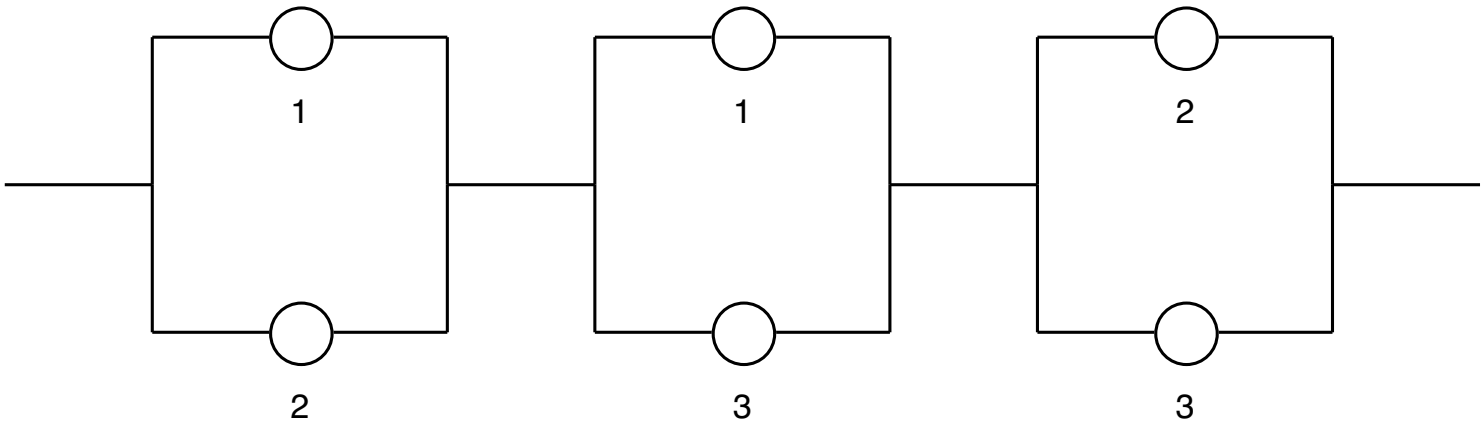


Figure: A reliability block diagram of a 2-out-of-3 system.

For a 2-out-of-3 system to fail 2 out of 3 components must fail. There are 3 possible subsets of components which contains 2 components: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$.



A 3-out-of-4 system

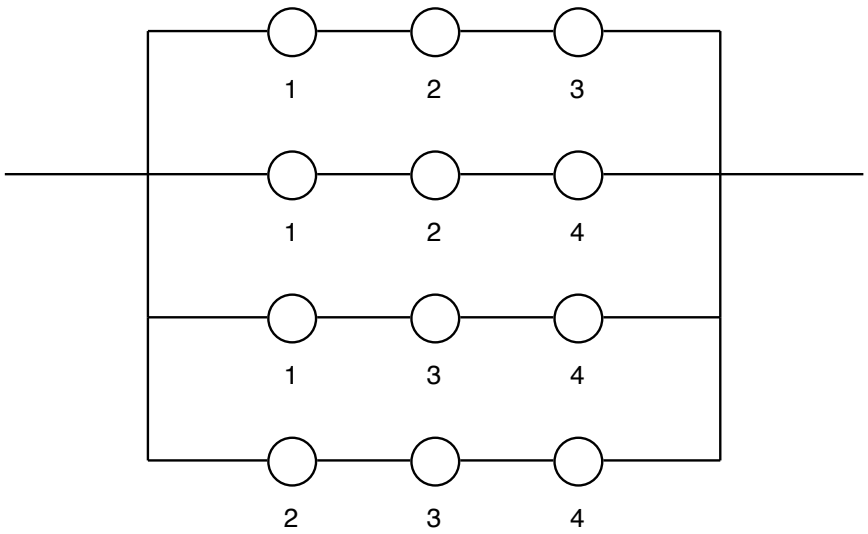


Figure: A reliability block diagram of a 3-out-of-4 system.

For a 3-out-of-4 system to function 3 out of 4 components must function. There are 4 possible subsets of components which contains 3 components: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$.



A 2-out-of-4 system

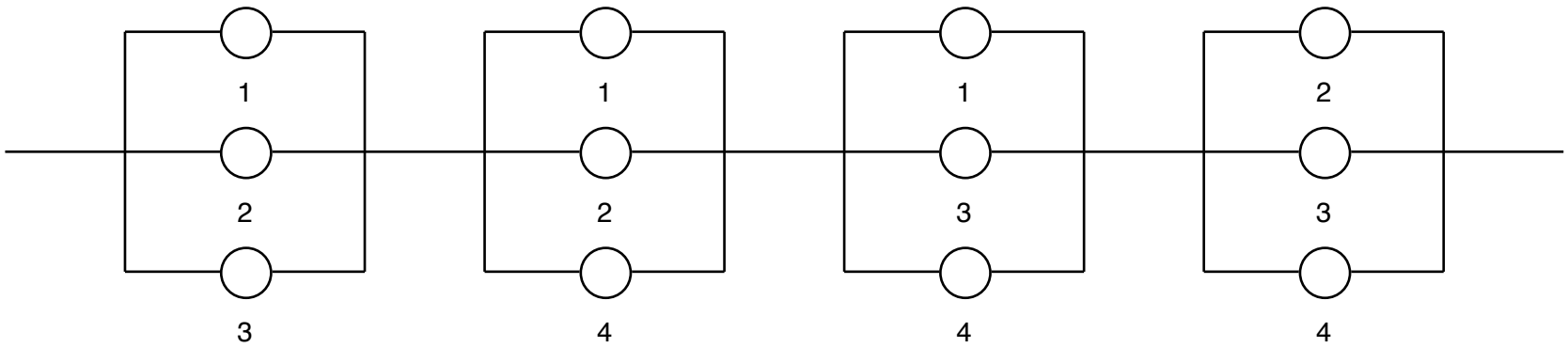


Figure: A reliability block diagram of a 2-out-of-4 system.

For a 2-out-of-4 system to fail 3 out of 4 components must fail. There are 4 possible subsets of components which contains 3 components: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$.



The reliability of a k-out-of-n system

In order to evaluate the reliability of a k-out-of-n system, we introduce the vector of stochastic component state variables $\boldsymbol{X} = (X_1, \ldots, X_n)$.

Moreover, it is convenient to introduce the following random variable:

$$S = \sum_{i=1}^{n} X_i.$$

We then get that:

$$h = P(\phi(\boldsymbol{X}) = 1) = P(\sum_{i=1}^{n} X_i \geq k) = P(S \geq k).$$



The reliability of a k-out-of-n system (cont.)

If the component states are *independent*, and the component reliabilities are all *equal*, i.e., $p_1 = \cdots = p_n = p$, the random variable S is a binomially distributed random variable, and we have:

$$P(S = i) = \binom{n}{i} p^i (1-p)^{n-i}.$$

Hence, the reliability of the system is given by:

$$h(\boldsymbol{p}) = h(p) = P(S \geq k) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$



The reliability of a 2-out-of-3 system

EXAMPLE: Let (C, ϕ) be a 2-out-of-3 system where the component states are independent, and where $p_1 = p_2 = p_3 = p$. We then have:

$$P(S = 2) = \binom{3}{2} p^2 (1-p)^1 = 3p^2(1-p)$$

$$P(S = 3) = \binom{3}{3} p^3 (1-p)^0 = p^3.$$

Hence, the reliability of the system is:

$$h = P(S \geq 2) = 3p^2(1-p) + p^3 = 3p^2 - 2p^3.$$



The reliability of a 3-out-of-4 system

EXAMPLE: Let (C, ϕ) be a 3-out-of-4 system where the component states are independent, and where $p_1 = p_2 = p_3 = p_4 = p$. We then have:

$$P(S = 3) = \binom{4}{3} p^3 (1-p)^1 = 4p^3(1-p)$$

$$P(S = 4) = \binom{4}{4} p^4 (1-p)^0 = p^4.$$

Hence, the reliability of the system is:

$$h = P(S \geq 3) = 4p^3(1-p) + p^4 = 4p^3 - 3p^4.$$



The reliability of a 2-out-of-3 system

When the component reliabilities are unequal, the reliability calculations are a bit more complicated. We will return to the general case in Chapter 4. Here we just consider two simple examples.

EXAMPLE: Let (C, ϕ) be a 2-out-of-3 system where the component states are independent with reliabilities p_1, p_2, p_3, We then have:

$$P(S = 2) = p_1 p_2 (1 - p_3) + p_1 (1 - p_2) p_3 + (1 - p_1) p_2 p_3$$

$$P(S = 3) = p_1 p_2 p_3$$

Hence, the reliability of the system is:

$$\begin{aligned} h &= p_1 p_2 (1 - p_3) + p_1 (1 - p_2) p_3 + (1 - p_1) p_2 p_3 + p_1 p_2 p_3 \\ &= p_1 p_2 + p_1 p_3 + p_2 p_3 - 2 p_1 p_2 p_3 . \end{aligned}$$



The reliability of a 3-out-of-4 system

EXAMPLE: Let (C, ϕ) be a 3-out-of-4 system where the component states are independent with reliabilities p_1, p_2, p_3, p_4, We then have:

$$P(S = 3) = p_1 p_2 p_3 (1 - p_4) + p_1 p_2 (1 - p_3) p_4 + p_1 (1 - p_2) p_3 p_4 + (1 - p_1) p_2 p_3 p_4$$

$$P(S = 4) = p_1 p_2 p_3 p_4$$

Hence, the reliability of the system is:

$$\begin{aligned} h &= p_1 p_2 p_3 (1 - p_4) + p_1 p_2 (1 - p_3) p_4 \\ &\quad + p_1 (1 - p_2) p_3 p_4 + (1 - p_1) p_2 p_3 p_4 \\ &\quad + p_1 p_2 p_3 p_4 \\ &= p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4 - 3 p_1 p_2 p_3 p_4 \end{aligned}$$

