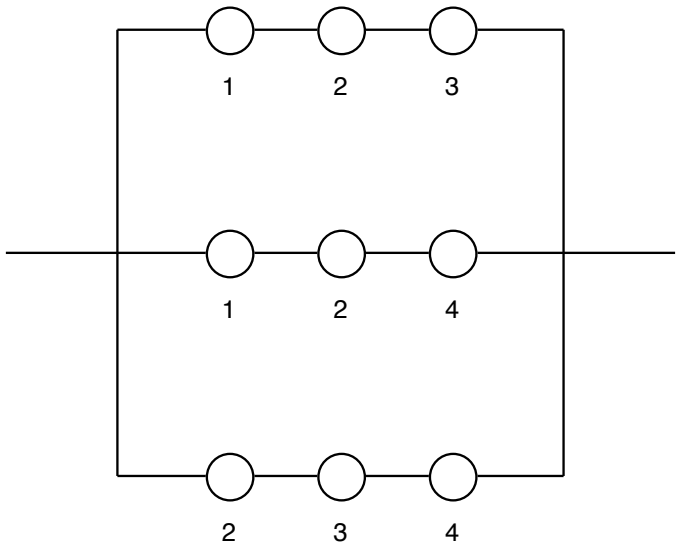
STK3405 – Lecture 23

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STK3405/4405 - Trial Exam, problem 1

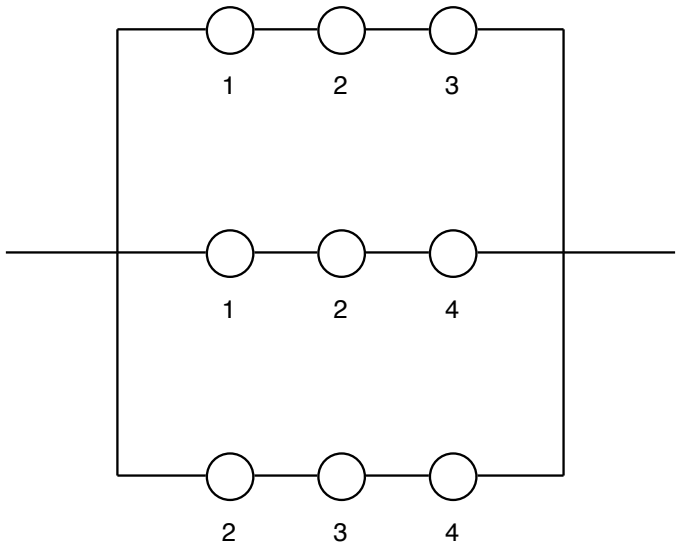


Consider the binary monotone system (C, ϕ) shown above. The component set of the system is $C = \{1, 2, 3, 4\}$.

Let $\boldsymbol{X} = (X_1, X_2, X_3, X_4)$ denote the vector of component state variables, and assume throughout this problem that X_1, X_2, X_3, X_4 are independent. Let $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, 3, 4$.



STK3405/4405 - Trial Exam, problem 1a



SOLUTION:

Minimal path sets: $\{1,2,3\}$, $\{1,2,4\}$, $\{2,3,4\}$.

Minimal cut sets: $\{2\}$, $\{1,3\}$, $\{1,4\}$, $\{3,4\}$

Since all components are contained in at least one minimal path set (or at least one minimal cut set), it follows that the system is *coherent*



STK3405/4405 - Trial Exam, problem 1b

Explain why the structure function of the system is given by:

$$\phi(\boldsymbol{X}) = X_2(X_1X_3 + X_1X_4 + X_3X_4 - 2X_1X_3X_4)$$

Describe briefly how this can be used to find $h(\boldsymbol{p}) = P(\phi(\boldsymbol{X}) = 1)$.

SOLUTION: We note that component 2 is *in series* with the rest of the system. This follows since component 2 is a member of all the minimal path sets. Alternatively, this follows since $\{2\}$ is a minimal cut set. As a result we have:

$$\phi(\boldsymbol{X}) = X_2 \cdot \phi(1_2, \boldsymbol{X})$$



STK3405/4405 - Trial Exam, problem 1b (cont.)

Given that component 2 is functioning, the minimal path sets are $\{1,3\}$, $\{1,4\}$, $\{3,4\}$. Thus, we have:

$$\begin{aligned}\phi(1_2, \boldsymbol{X}) &= (X_1 X_3) \amalg (X_1 X_4) \amalg (X_3 X_4) \\ &= 1 - (1 - X_1 X_3)(1 - X_1 X_4)(1 - X_3 X_4) \\ &= X_1 X_3 + X_1 X_4 + X_3 X_4 - 2X_1 X_3 X_4\end{aligned}$$

Hence, we get that:

$$\begin{aligned}\phi(\boldsymbol{X}) &= X_2 \cdot \phi(1_2, \boldsymbol{X}) \\ &= X_2(X_1 X_3 + X_1 X_4 + X_3 X_4 - 2X_1 X_3 X_4)\end{aligned}$$



STK3405/4405 - Trial Exam, problem 1b (cont.)

Since the component state variables X_1, X_2, X_3, X_4 are assumed to be independent, the reliability of the system is:

$$
\begin{aligned}
h(\boldsymbol{p}) &= P(\phi(\boldsymbol{X}) = 1) = E[\phi(\boldsymbol{X})] \\
&= E[X_2(X_1X_3 + X_1X_4 + X_3X_4 - 2X_1X_3X_4)] \\
&= E[X_2](E[X_1]E[X_3] + E[X_1]E[X_4] + E[X_3]E[X_4] - 2E[X_1]E[X_3]E[X_4]) \\
&= p_2(p_1p_3 + p_1p_4 + p_3p_4 - 2p_1p_3p_4)
\end{aligned}
$$



STK3405/4405 - Trial Exam, problem 1c

The Birnbaum measure of the *structural importance* of component $i \in C$ is defined as:

$$J_B^{(i)} = \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})],$$

where n is the number of components in the system. Give an interpretation of $J_B^{(i)}$ and discuss possible applications of this measure.

SOLUTION: We start out by noting that:

$$\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x}) = 1$$

if and only if $(\cdot_i, \boldsymbol{x})$ is a *critical* vector for component i. Thus, the sum:

$$\sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})]$$

is equal to the number of *critical* vectors for component i.



STK3405/4405 - Trial Exam, problem 1c (cont.)

Since the total number of binary vectors $(\cdot_i, \boldsymbol{x})$ is equal to 2^{n-1}, it follows that $J_B^{(i)}$ can be interpreted as the fraction of binary vectors $(\cdot_i, \boldsymbol{x})$ where component i is critical.

The structural importance measure can be used to identify the *most critical components* in a binary monotone system. The measure does not take into account the *reliabilities* of the components. Thus, the measure can be calculated in an *early state* of the design process before specific component choices have been made.

By identifying the most critical components one can choose to invest more in these particular components in order to maximize the system reliability.

Alternatively, if a component has a very high structural importance and this component is very expensive, it may be better to introduce more *redundancy* into the design and thus reduce the structural importance of this particular component



STK3405/4405 - Trial Exam, problem 1d

Without doing any calculations, rank the components in (C, ϕ) with respect to their respective structural importance. Explain your answer.

SOLUTION: Since component 2 is in series with the rest of the system, and none of the other components are in series with the rest of the system, we know that:

$$J_B^{(2)} > J_B^{(i)}, \quad i = 1, 3, 4.$$

Furthermore, by symmetry it follows that:

$$J_B^{(1)} = J_B^{(3)} = J_B^{(4)}$$

Thus, component 2 is the most important component, while the other components are equally important



STK3405/4405 - Trial Exam, problem 1e

The Birnbaum measure of the *reliability importance* of component $i \in C$ is denoted by $I_B^{(i)}$. Describe briefly how $I_B^{(i)}$ is defined, and discuss the relation between $I_B^{(i)}$ and $J_B^{(i)}$.

SOLUTION: The Birnbaum measure of the *reliability importance* of component $i \in C$, denoted by $I_B^{(i)}$, is defined as:

$$\begin{aligned} I_B^{(i)} &= P(\text{Component } i \text{ is critical}) = P(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1) \\ &= E[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})] \end{aligned}$$

If the component state variables are *independent* and $P(X_i = 1) = \frac{1}{2}$ for all $i \in C$, the joint distribution of $(\cdot_i, \boldsymbol{X})$ is given by:

$$P\big((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})\big) = \frac{1}{2^{n-1}}, \qquad \text{for all } (\cdot_i, \boldsymbol{x})$$



STK3405/4405 - Trial Exam, problem 1e (cont.)

Hence, we get:

$$I_B^{(i)} = E[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})]$$

$$= \sum_{(\cdot_i, \boldsymbol{X})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})] \cdot P\big((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})\big)$$

$$= \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{X})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})] = J_B^{(i)}$$

Thus, the structural importance measure is a special case of the reliability importance measure corresponding to the case where component state variables are *independent* and where $P(X_i = 1) = \frac{1}{2}$ for all $i \in C$



STK3405/4405 - Trial Exam, problem 2a

Let (C, ϕ) be a threshold system with structure function:

$$\phi(\boldsymbol{x}) = I(\sum_{i=1}^{n} a_i x_i \geq b),$$

where $a_1, \ldots, a_n$ and b are non-negative real numbers.

Show that the dual of this system is a threshold system as well.

SOLUTION: The dual structure function is given by:

$$\phi^D(\boldsymbol{y}) = 1 - \phi(\mathbf{1} - \boldsymbol{y}) = 1 - I(\sum_{i=1}^{n} a_i(1 - y_i) \geq b)$$

$$= 1 - I(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i y_i \geq b) = 1 - I(\sum_{i=1}^{n} a_i y_i \leq \sum_{i=1}^{n} a_i - b)$$



STK3405/4405 - Trial Exam, problem 2a (cont.)

Furthermore:

$$\phi^D(\boldsymbol{y}) \;=\; 1 - I(\sum_{i=1}^{n} a_i y_i \leq \sum_{i=1}^{n} a_i - b) \;=\; I(\sum_{i=1}^{n} a_i y_i > \sum_{i=1}^{n} a_i - b)$$

Since the set of possible values of $\sum_{i=1}^{n} a_i y_i$ is finite, we may define:

$$b^D = \min\{\sum_{i=1}^{n} a_i y_i : \sum_{i=1}^{n} a_i y_i > \sum_{i=1}^{n} a_i - b\}$$

Hence, it follows that:

$$\phi^D(\boldsymbol{y}) \;=\; I(\sum_{i=1}^{n} a_i y_i > \sum_{i=1}^{n} a_i - b) \;=\; I(\sum_{i=1}^{n} a_i y_i \geq b^D)$$

completing the proof that the dual system is a threshold system as well.



STK3405/4405 - Trial Exam, problem 2b

Let $j \in C$, and consider the minors $(C \setminus j, \phi(1_j, \cdot))$ and $(C \setminus j, \phi(0_j, \cdot))$.

Show that $(C \setminus j, \phi(1_j, \cdot))$ and $(C \setminus j, \phi(0_j, \cdot))$ are threshold systems as well.

SOLUTION: We observe that:

$$\phi(1_j, \boldsymbol{x}) = I(a_j \cdot 1 + \sum_{i \neq j} a_i x_i \geq b) = I(\sum_{i \neq j} a_i x_i \geq b - a_j)$$

$$\phi(0_j, \boldsymbol{x}) = I(a_j \cdot 0 + \sum_{i \neq j} a_i x_i \geq b) = I(\sum_{i \neq j} a_i x_i \geq b)$$

Hence, it follows that $(C \setminus j, \phi(1_j, \cdot))$ and $(C \setminus j, \phi(0_j, \cdot))$ are threshold systems as well.



STK3405/4405 - Trial Exam, problem 2c

Assume that $n = 4$, $a_1 = a_2 = a_3 = a_4 = 1$, and $b = 2$.

(c) What kind of system is (C, ϕ) in this case?

SOLUTION: In this case the structure function ϕ simplifies to:

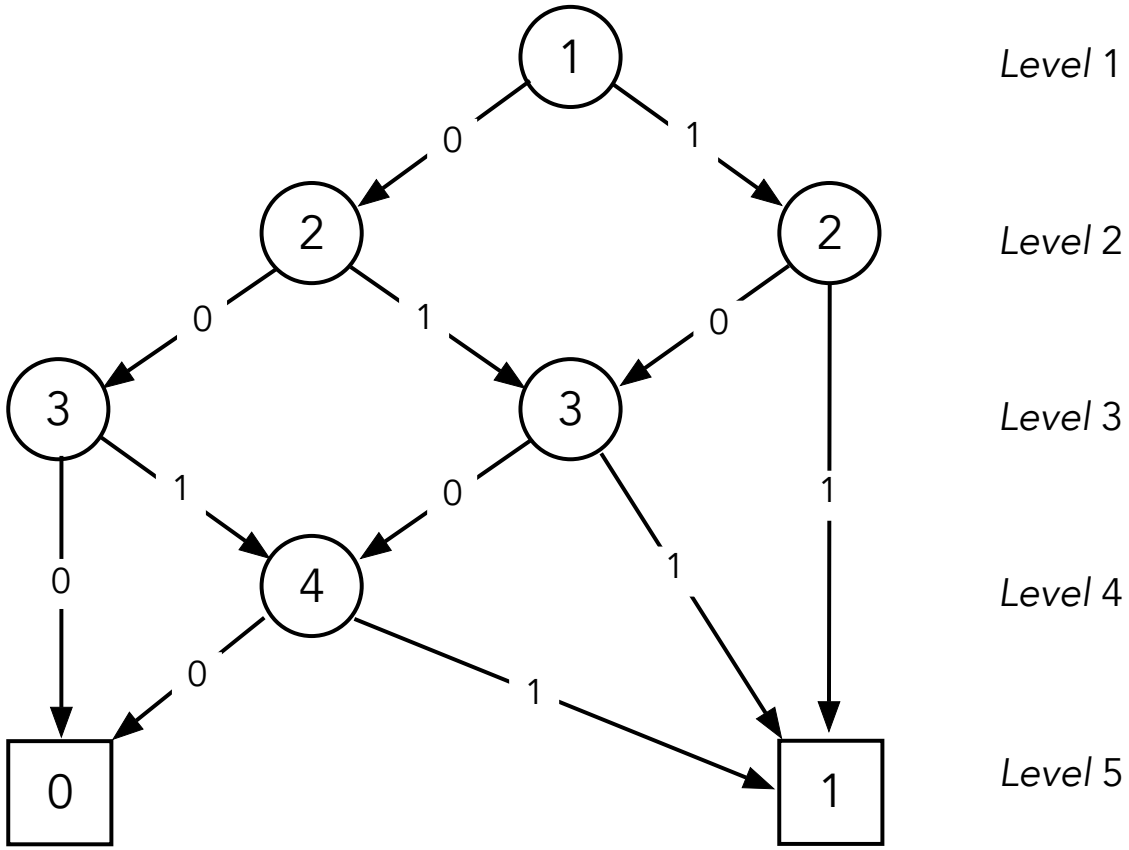
$$\phi(\boldsymbol{x}) = I(\sum_{i=1}^{n} a_i x_i \geq b) = I(\sum_{i=1}^{n} x_i \geq 2)$$

Thus, ϕ is the structure function of a 2-out-of-4 system.



STK3405/4405 - Trial Exam, problem 2d

Construct a reduced ordered binary decision diagram of (C, ϕ) in this case.





STK3405/4405 - Trial Exam, problem 3a

Assume that T is Weibull distributed with parameters $\alpha > 0$ and $\beta > 0$. Thus, the *survival function* of T is given by:

$$\bar{F}_T(t) = P(T > t) = \begin{cases} e^{-(t/\beta)^\alpha}, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

Show that the density of T is given by:

$$f_T(t) = \begin{cases} \frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1} e^{-(t/\beta)^\alpha}, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

SOLUTION: We focus on the case where $t \geq 0$. The result then follows since:

$$f_T(t) = -\frac{d}{dt}\bar{F}_T(t) = -e^{-(t/\beta)^\alpha} \cdot (-\alpha)\left(\frac{t}{\beta}\right)^{\alpha-1} \cdot \frac{1}{\beta}$$



STK3405/4405 - Trial Exam, problem 3b

Show that:

$$E[T] = \beta \cdot \Gamma(\frac{1}{\alpha} + 1),$$

where:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx, \quad a > 0.$$

Hint: Use that if X is a non-negative absolutely continuous random variable with survival function $\bar{F}_X(x) = P(X > x)$, then:

$$E[X] = \int_0^\infty \bar{F}_X(x) dx$$

as well as that $a \cdot \Gamma(a) = \Gamma(a+1)$, for all $a > 0$.



STK3405/4405 - Trial Exam, problem 3b

SOLUTION: We recall that if T is Weibull distributed with parameters α and β, then:

$$P(T > t) = e^{-(t/\beta)^{\alpha}}, \quad t \geq 0.$$

Hence, by using the first hint we get:

$$E[T] = \int_0^{\infty} \bar{F}_T(t)dt = \int_0^{\infty} e^{-(t/\beta)^{\alpha}} dt$$

Substitute $u = (t/\beta)^{\alpha}$, and $du = \frac{\alpha}{\beta}(t/\beta)^{\alpha-1}dt = \frac{\alpha}{\beta}u^{\frac{\alpha-1}{\alpha}}dt = \frac{\alpha}{\beta}u^{1-\frac{1}{\alpha}}dt$:

$$E[T] = \frac{\beta}{\alpha}\int_0^{\infty} u^{\frac{1}{\alpha}-1}e^{-u}du = \beta \cdot \frac{1}{\alpha} \cdot \Gamma(\frac{1}{\alpha}) = \beta \cdot \Gamma(\frac{1}{\alpha}+1)$$

where the last equation follows by the second hint.



STK3405/4405 - Trial Exam, problem 3c

Show that the median of T is given by:

$$Med[T] = \beta \cdot [\ln(2)]^{1/\alpha}$$

SOLUTION: $Med[T]$ is the solution to the equation:

$$\bar{F}_T(Med[T]) = e^{-(Med[T]/\beta)^{\alpha}} = \frac{1}{2}$$

This equation is equivalent to:

$$-(Med[T]/\beta)^{\alpha} = -\ln(2)$$

which has the solution:

$$Med[T] = \beta \cdot [\ln(2)]^{1/\alpha}$$



STK3405/4405 - Trial Exam, problem 3d

Show that the failure rate function of T is given by:

$$r_T(t) = \begin{cases} \frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1}, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

Under which conditions does T have an increasing failure rate?

SOLUTION: We focus on the case where $t \geq 0$. The result then follows since:

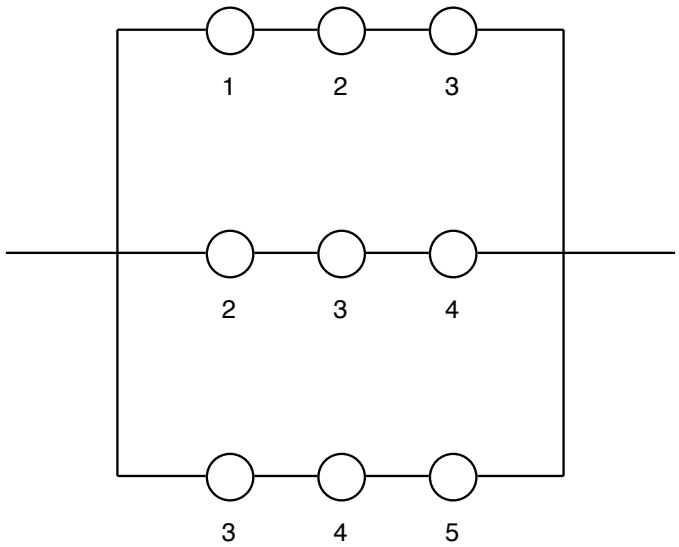
$$r_T(t) = \frac{f_T(t)}{\bar{F}(t)} = \frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1}$$

We see that $r_T(t)$ is increasing when $\alpha > 1$ and decreasing when $0 < \alpha < 1$.

In the special case where $\alpha = 1$, we get $r_T(t) = \beta^{-1}$ which is constant.



STK3405/4405 - Exam 2022, problem 1



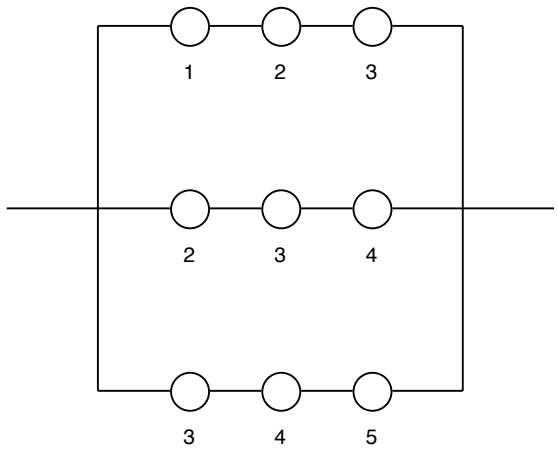
Consider the binary monotone system (C, ϕ) shown above. The component set of the system is $C = \{1, 2, 3, 4, 5\}$.

Let $\boldsymbol{X} = (X_1, X_2, X_3, X_4, X_5)$ denote the vector of component state variables, and assume throughout this problem that X_1, X_2, X_3, X_4, X_5 are independent. Let $\boldsymbol{p} = (p_1, p_2, p_3, p_4, p_5)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, 3, 4, 5$.



STK3405/4405 - Exam 2022, problem 1a

a) Find the minimal path sets (3 sets) and the minimal cut sets (4 sets) of the system. Is the system *coherent*?



SOLUTION: Minimal path sets: $P_1 = \{1, 2, 3\}$, $P_2 = \{2, 3, 4\}$, $P_3 = \{3, 4, 5\}$.
Minimal cut sets: $K_1 = \{1, 4\}$, $K_2 = \{2, 4\}$, $K_3 = \{2, 5\}$, $K_4 = \{3\}$.

Since $P_1 \cup P_2 \cup P_3 = C$, all components are contained in at least one minimal path set. Hence, it follows that the system is *coherent*.



STK3405/4405 - Exam 2022, problem 1b

b) Show that the structure function of the system is given by:

$$\phi(\boldsymbol{X}) = X_1X_2X_3 + X_2X_3X_4 + X_3X_4X_5 - X_1X_2X_3X_4 - X_2X_3X_4X_5$$

Compute $h(\boldsymbol{p})$ given that $\boldsymbol{p} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

SOLUTION: Using the minimal path sets we get:

$$\phi(\boldsymbol{X}) = \coprod_{j=1}^{3} \prod_{i \in P_j} X_i = 1 - (1 - X_1X_2X_3)(1 - X_2X_3X_4)(1 - X_3X_4X_5)$$

$$\begin{aligned} &= X_1X_2X_3 + X_2X_3X_4 + X_3X_4X_5 \\ &- X_1X_2X_3X_4 - X_2X_3X_4X_5 - X_1X_2X_3X_4X_5 + X_1X_2X_3X_4X_5 \end{aligned}$$

$$= X_1X_2X_3 + X_2X_3X_4 + X_3X_4X_5 - X_1X_2X_3X_4 - X_2X_3X_4X_5$$



STK3405/4405 - Exam 2022, problem 1b (cont.)

The reliability of the system is then found by calculating $E[\phi(\boldsymbol{X})]$:

$$
\begin{aligned}
h(\boldsymbol{p}) &= P(\phi(\boldsymbol{X}) = 1) = E[\phi(\boldsymbol{X})] \\
&= E[X_1]E[X_2]E[X_3] + E[X_2]E[X_3]E[X_4] + E[X_3]E[X_4]E[X_5] \\
&\quad - E[X_1]E[X_2]E[X_3]E[X_4] - E[X_2]E[X_3]E[X_4]E[X_5] \\
&= p_1p_2p_3 + p_2p_3p_4 + p_3p_4p_5 - p_1p_2p_3p_4 - p_2p_3p_4p_5
\end{aligned}
$$

We then insert $p_i = \frac{1}{2}$ and get:

$$
h(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = 3 \cdot \tfrac{1}{8} - 2 \cdot \tfrac{1}{16} = \tfrac{3-1}{8} = \tfrac{1}{4}.
$$



STK3405/4405 - Exam 2022, problem 1c

The Birnbaum measure of the *reliability importance* of component i is defined as:

$$I_B^{(i)} = P(\text{Component } i \text{ is critical for the system}), \quad i \in C.$$

c) Show that:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \quad i \in C.$$

SOLUTION: Using pivotal decomposition we have:

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p})$$



STK3405/4405 - Exam 2022, problem 1c (cont.)

We then take the partial derivative of $h(\boldsymbol{p})$ with respect to p_i and get:

$$\begin{aligned}\frac{\partial h(\boldsymbol{p})}{\partial p_i} &= h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p}) \\ &= E[\phi(1_i, \boldsymbol{X})] - E[\phi(0_i, \boldsymbol{X})] = E[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})] \\ &= P(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1) \\ &= P(\text{Component } i \text{ is critical for the system}) = I_B^{(i)},\end{aligned}$$

where the second equality follows since the component state variables are assumed to be stochastically independent.



STK3405/4405 - Exam 2022, problem 1d

d) Use the result from (c) to show that:

$$I_B^{(1)} = p_2 p_3 (1 - p_4)$$
$$I_B^{(2)} = p_1 p_3 (1 - p_4) + p_3 p_4 (1 - p_5)$$
$$I_B^{(3)} = p_1 p_2 (1 - p_4) + p_2 p_4 (1 - p_5) + p_4 p_5$$

SOLUTION: From (b) we know that:

$$h(\boldsymbol{p}) = p_1 p_2 p_3 + p_2 p_3 p_4 + p_3 p_4 p_5 - p_1 p_2 p_3 p_4 - p_2 p_3 p_4 p_5$$

Hence:

$$I_B^{(1)} = \partial h(\boldsymbol{p}) / \partial p_1 = p_2 p_3 - p_2 p_3 p_4 = p_2 p_3 (1 - p_4)$$

$$\begin{aligned} I_B^{(2)} &= \partial h(\boldsymbol{p}) / \partial p_2 = p_1 p_3 + p_3 p_4 - p_1 p_3 p_4 - p_3 p_4 p_5 \\ &= p_1 p_3 (1 - p_4) + p_3 p_4 (1 - p_5) \end{aligned}$$

$$\begin{aligned} I_B^{(3)} &= \partial h(\boldsymbol{p}) / \partial p_3 = p_1 p_2 + p_2 p_4 + p_4 p_5 - p_1 p_2 p_4 - p_2 p_4 p_5 \\ &= p_1 p_2 (1 - p_4) + p_2 p_4 (1 - p_5) + p_4 p_5 \end{aligned}$$



STK3405/4405 - Exam 2022, problem 1e

e) Assume that $p_1 = p_2 = p_3 = p_4 = p_5 = p$, where $0 < p < 1$. Show that:

$$I_B^{(1)} = I_B^{(5)} < I_B^{(2)} = I_B^{(4)} < I_B^{(3)}$$

SOLUTION: Using the results from (d) we get:

$$I_B^{(1)} = p^2(1-p)$$

$$I_B^{(2)} = 2p^2(1-p) = 2 \cdot I_B^{(1)}$$

$$I_B^{(3)} = 2p^2(1-p) + p^2 = I_B^{(2)} + p^2 = 2 \cdot I_B^{(1)} + p^2$$

By symmetry, we have:

$$I_B^{(5)} = I_B^{(1)} \quad \text{and} \quad I_B^{(4)} = I_B^{(2)}$$

Hence, we get the following ranking:

$$I_B^{(1)} = I_B^{(5)} < I_B^{(2)} = I_B^{(4)} < I_B^{(3)}$$



STK3405/4405 - Exam 2022, problem 2a

Let (C, ϕ) be a series system of non-repairable components where $C = \{1, \ldots, n\}$, and let for $t \geq 0$ and $i \in C$:

$$X_i(t) = \begin{cases} 1 & \text{if component } i \text{ is functioning at time } t \\ 0 & \text{otherwise} \end{cases}$$

Moreover, let $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$. We assume that the component processes $\{X_1(t)\}_{t\geq 0}, \ldots, \{X_n(t)\}_{t\geq 0}$ are stochastically independent.

The lifetime of component i is denoted by T_i, and we assume that the survival function of T_i is $\bar{F}_i$, for $i \in C$.

Since the component processes are assumed to be stochastically independent, it follows that $T_1, \ldots, T_n$ are stochastically independent as well, and we have:

$$P(T_i > t) = \bar{F}_i(t), \quad i \in C, \quad t \geq 0,$$



STK3405/4405 - Exam 2022, problem 2a (cont.)

a) Show that:

$$P(\phi(\boldsymbol{X}(t)) = 1) = \prod_{i=1}^{n} \bar{F}_i(t), \quad t \geq 0.$$

SOLUTION: We note that the event $\{X_i(t) = 1\}$ is equivalent to the event $\{T_i > t\}$, for $t \geq 0$ and $i \in C$. Hence, since the component processes are stochastically independent, we get for $t \geq 0$:

$$\begin{aligned} P(\phi(\boldsymbol{X}(t)) = 1) &= P(\prod_{i=1}^{n} X_i = 1) = \prod_{i=1}^{n} P(X_i(t) = 1) \\ &= \prod_{i=1}^{n} P(T_i > t) = \prod_{i=1}^{n} \bar{F}_i(t). \end{aligned}$$



STK3405/4405 - Exam 2022, problem 2b

The *time dependent* Birnbaum measure of *reliability importance* of component $i \in C$ at time $t \geq 0$ is defined as:

$$I_B^{(i)}(t) = P(\text{Component } i \text{ is critical for the system at time } t)$$

b) Show that:

$$I_B^{(i)}(t) = \prod_{j \neq i} \bar{F}_j(t), \quad i \in C, \quad t \geq 0.$$



STK3405/4405 - Exam 2022, problem 2b (cont.)

SOLUTION: By the definition of the Birnbaum measure we get:

$$
\begin{aligned}
I_B^{(i)}(t) &= P(\text{Component } i \text{ is critical for the system at time } t) \\
&= E[\phi(1_i, \boldsymbol{X}(t))] - E[\phi(0_i, \boldsymbol{X}(t))] \\
&= P(\prod_{j \neq i} X_j(t) = 1) - 0 = \prod_{j \neq i} P(X_j(t) = 1) \\
&= \prod_{j \neq i} P(T_j > t) = \prod_{j \neq i} \bar{F}_j(t)
\end{aligned}
$$



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The Barlow-Proschan measure of *reliability importance* of component $i \in C$ is defined as:

$$I_{B-P}^{(i)} = P(\text{Component } i \text{ fails at the same time as the system})$$

Assuming that $T_1, \ldots, T_n$ are absolutely continuously distributed with densities $f_1, \ldots, f_n$ respectively, it can be shown that we have:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) \cdot f_i(t) dt$$

Finally, we assume that:

$$\bar{F}_i(t) = e^{-\lambda_i t^\alpha}, \quad i \in C, \quad t \geq 0.$$

where $\alpha > 0$ and $\lambda_i > 0$, for all $i \in C$.

c) Show that:

$$I_{B-P}^{(i)} = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}, \quad i \in C.$$



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SOLUTION: We let $R(t) = t^{\alpha}$ and note that since $\alpha > 0$, $R(t)$ is increasing, $R(0) = 0$ and $R(\infty) = \infty$. The survival functions of the lifetimes $T_1, \ldots, T_n$ can then be expressed as:

$$\bar{F}_i(t) = e^{-\lambda_i R(t)}, \quad i \in C, \quad t \geq 0.$$

Moreover, the densities of the lifetimes $T_1, \ldots, T_n$ can then be expressed as:

$$f_i(t) = -\frac{d\bar{F}_i(t)}{dt} = \lambda_i R'(t) \exp(-\lambda_i R(t)) = \lambda_i R'(t) \bar{F}_i(t)$$

Furthermore, by the result in (b) we have:

$$I_B^{(i)}(t) = \prod_{j \neq i} \bar{F}_j(t)$$



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Hence, the Barlow-Proschan measure becomes:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt = \int_0^\infty \prod_{j \neq i} \bar{F}_j(t) \lambda_i R'(t) \bar{F}_i(t) dt$$

$$= \int_0^\infty \prod_{j=1}^n \bar{F}_j(t) \lambda_i R'(t) dt = \lambda_i \int_0^\infty e^{-\sum_{j=1}^n \lambda_j R(t)} R'(t) dt$$

$$= \lambda_i \left[-\frac{1}{\sum_{j=1}^n \lambda_j} e^{-\sum_{j=1}^n \lambda_j R(t)} \right]_{t=0}^\infty = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}, \quad i \in C,$$

where the last equality follows since $R(0) = 0$ and $R(\infty) = \infty$ and $\lambda_i > 0$, for all $i \in C$.



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ALT. SOLUTION: The problem can also be solved as follows:

We start out by finding the densities of the lifetimes $T_1, \ldots, T_n$:

$$f_i(t) = -\frac{d\bar{F}_i(t)}{dt} = \lambda_i \alpha t^{\alpha-1} e^{-\lambda_i t^\alpha} = \lambda_i \alpha t^{\alpha-1} \bar{F}_i(t)$$

Furthermore, by the result in (b) we have:

$$I_B^{(i)}(t) = \prod_{j \neq i} \bar{F}_j(t)$$

Hence, the Barlow-Proschan measure becomes:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt = \int_0^\infty \prod_{j \neq i} \bar{F}_j(t) \lambda_i \alpha t^{\alpha-1} \bar{F}_i(t) dt$$

$$= \int_0^\infty \prod_{j=1}^n \bar{F}_j(t) \lambda_i \alpha t^{\alpha-1} dt = \lambda_i \int_0^\infty \alpha t^{\alpha-1} e^{-\sum_{j=1}^n \lambda_j t^\alpha} dt$$



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We then observe that the integrand of the last integral:

$$\lambda_i \int_0^\infty \alpha t^{\alpha-1} e^{-\sum_{j=1}^n \lambda_j t^\alpha} dt$$

is *almost* a density function of the same type as above.

In order to get a proper density, we need to multiply it by the normalising factor $\sum_{i=1}^n \lambda_j$.

At the same time we divide by the same factor outside the integral. The resulting integral then becomes an integral of a density over its outcome space $[0, \infty)$, which is equal to one. Hence:

$$I_{B-P}^{(i)} = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j} \int_0^\infty \sum_{j=1}^n \lambda_j \alpha t^{\alpha-1} e^{-\sum_{j=1}^n \lambda_j t^\alpha} dt = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}, \quad i \in C.$$

