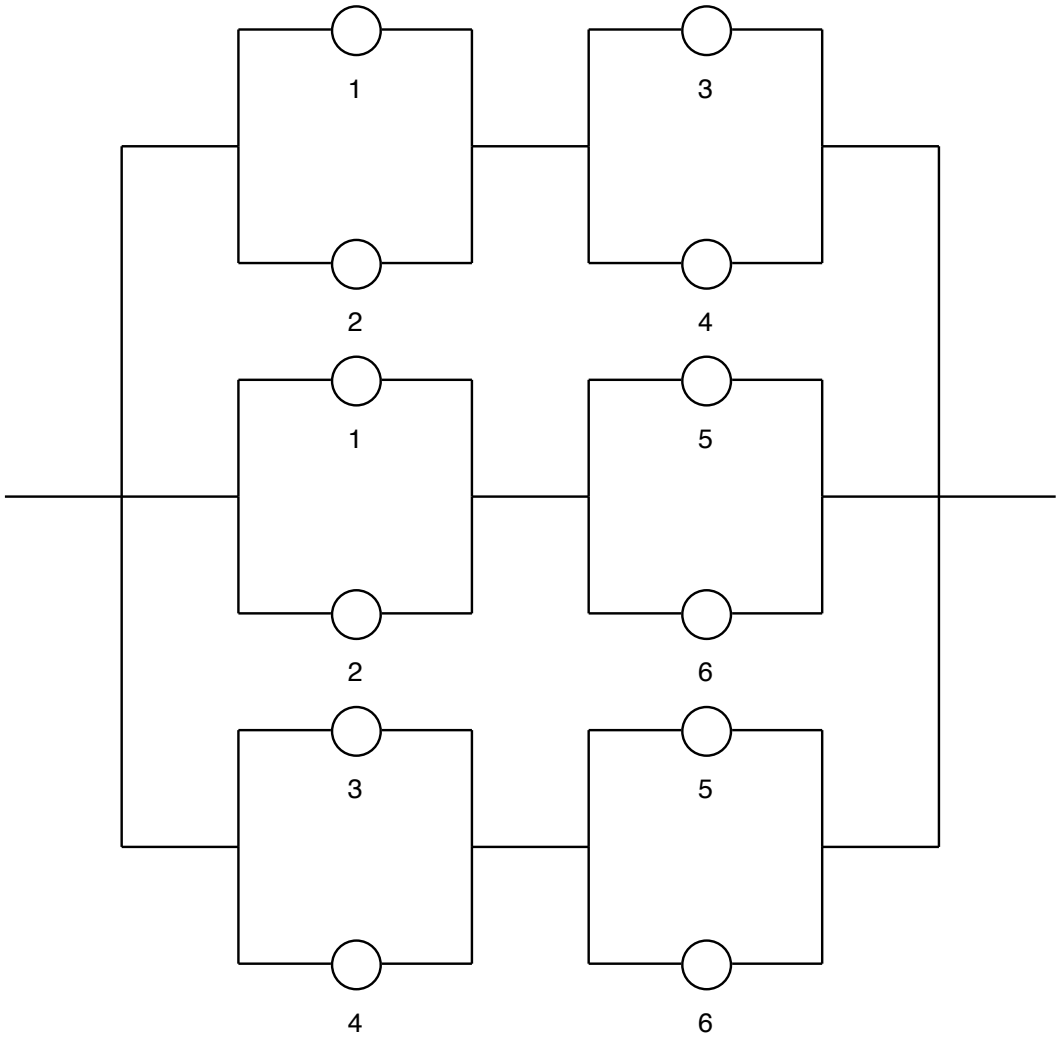
STK3405 – Lecture 24

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STK3405/4405 - Exam 2023, problem 1





STK3405/4405 - Exam 2023, problem 1

Consider the binary monotone system (C, ϕ) shown on the previous slide, where:

$$C = \{1, \ldots, 6\} \quad \text{is the component set.}$$

Moreover, we let:

$$\boldsymbol{X} = (X_1, \ldots, X_6) \quad \text{The vector of component state variables.}$$

We assume throughout this problem that $X_1, \ldots, X_6$ are independent, and that:

$$P(X_i = 1) = p, \quad i = 1, \ldots, 6.$$



STK3405/4405 - Exam 2023, problem 1a

a) Find the minimal path sets (12 sets) and the minimal cut sets (3 sets) of the system. Is the system *coherent*?

SOLUTION:

Minimal path sets:

$$P_1 = \{1,3\},\ P_2 = \{1,4\},\ P_3 = \{2,3\},\ P_4 = \{2,4\}$$
$$P_5 = \{1,5\},\ P_6 = \{1,6\},\ P_7 = \{2,5\},\ P_8 = \{2,6\}$$
$$P_9 = \{3,5\},\ P_{10} = \{3,6\},\ P_{11} = \{4,5\},\ P_{12} = \{4,6\}.$$

Minimal cut sets:

$$K_1 = \{1,2,3,4\},\ K_2 = \{1,2,5,6\},\ K_3 = \{3,4,5,6\}$$

Since $P_1 \cup \ldots \cup P_{12} = C$, all components are contained in at least one minimal path set. Hence, it follows that the system is *coherent*.

Alternatively, since $K_1 \cup K_2 \cup K_3 = C$, all components are contained in at least one minimal cut set. Hence, it follows that the system is *coherent* ■



STK3405/4405 - Exam 2023, problem 1b

We introduce the modules (A_1, χ_1), (A_2, χ_2) and (A_3, χ_3), where $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$ and $A_3 = \{5, 6\}$, and where:

$$\chi_1(X_1, X_2) = X_1 \amalg X_2$$
$$\chi_2(X_3, X_4) = X_3 \amalg X_4$$
$$\chi_3(X_5, X_6) = X_5 \amalg X_6$$

b) Explain why the structure function of the system is given by:

$$\phi(\boldsymbol{X}) = \mathsf{I}(\chi_1(X_1, X_2) + \chi_2(X_3, X_4) + \chi_3(X_5, X_6) \geq 2)$$

where $\mathsf{I}(\cdot)$ denotes the indicator function.



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SOLUTION: We observe that the system can be interpreted as a 2-out-of-3 system, but where the components are the modules (A_1, χ_1), (A_2, χ_2) and (A_3, χ_3).

The structure function of a 2-out-of-3 system can be written as the indicator function of the event that the sum of the component states is greater than or equal to 2. Hence, it follows that:

$$\phi(\boldsymbol{X}) = \mathsf{I}(\chi_1(X_1, X_2) + \chi_2(X_3, X_4) + \chi_3(X_5, X_6) \geq 2)$$

as claimed



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c) Show that:

$$P(\chi_i = 1) = 2p - p^2, \quad i = 1, 2, 3.$$

SOLUTION: Since $P(X_i = 1) = p$ for all $i \in C$, it follows that:

$$\begin{aligned} P(\chi_1(X_1, X_2) = 1) &= \mathrm{E}[\chi_1(X_1, X_2)] = \mathrm{E}[X_1 \amalg X_2] \\ &= \mathrm{E}[1 - (1 - X_1)(1 - X_2)] = \mathrm{E}[X_1 + X_2 - X_1 X_2] \\ &= \mathrm{E}[X_1] + \mathrm{E}[X_2] - \mathrm{E}[X_1]\,\mathrm{E}[X_2] \\ &= p + p - p \cdot p = 2p - p^2. \end{aligned}$$

By the same argument we also get that:

$$P(\chi_2(X_3, X_4) = 1) = \cdots = 2p - p^2$$
$$P(\chi_3(X_5, X_6) = 1) = \cdots = 2p - p^2$$



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We then let:

$$S = S(\boldsymbol{X}) = \chi_1(X_1, X_2) + \chi_2(X_3, X_4) + \chi_3(X_5, X_6)$$

d) Show that:

$$P(S = s) = \binom{3}{s}(2p - p^2)^s(1 - 2p + p^2)^{3-s}, \quad s = 0, 1, 2, 3.$$

SOLUTION: We observe that S is the sum three independent and identically distributed binary variables. Hence, S is binomially distributed.

By (c) we know that $P(\chi_i = 1) = 2p - p^2$ for $i = 1, 2, 3$. Hence, it follows that $S \sim Bin(3, 2p - p^2)$. Thus, we have:

$$P(S = s) = \binom{3}{s}(2p - p^2)^s(1 - 2p + p^2)^{3-s}, \quad s = 0, 1, 2, 3.$$



STK3405/4405 - Exam 2023, problem 1e

e) Show that:

$$P(\phi = 1) = p^2(2-p)^2(3-4p+2p^2)$$

SOLUTION: We have that:

$$\begin{aligned}
P(\phi = 1) &= P(S \geq 2) = P(S = 2) + P(S = 3) \\
&= \binom{3}{2}(2p - p^2)^2(1 - 2p + p^2) + \binom{3}{3}(2p - p^2)^3 \\
&= 3(2p - p^2)^2(1 - 2p + p^2) + (2p - p^2)^3 \\
&= (2p - p^2)^2(3 - 6p + 3p^2 + 2p - p^2) \\
&= p^2(2-p)^2(3-4p+2p^2)
\end{aligned}$$



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We denote the minimal path sets by $P_1, \ldots, P_{12}$, and introduce the events:

$$E_j = \{\text{All of the components in } P_j \text{ are functioning}\}, \quad j = 1, \ldots, 12.$$

f) Explain briefly why:

$$P(\phi = 1) = P(\bigcup_{j=1}^{12} E_j)$$

Assuming that we calculate the probability of this union of events using the *inclusion-exclusion* formula, how many terms would this formula contain before simplification?



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SOLUTION: The system is functioning if and only if at least one minimal path set is functioning.

A minimal path set is functioning if and only if all the components in that set are functioning.

Since the event E_j is the event that all of the components in P_j are functioning, $j = 1, \ldots, 12$, the event that the system is functioning is equivalent to the union of the events $E_1, \ldots, E_{12}$.

Hence, the probability that the system is functioning is equal to the probability of this union:

$$P(\phi = 1) = P(\bigcup_{j=1}^{12} E_j)$$



STK3405/4405 - Exam 2023, problem 1f (cont.)

By the inclusion-exclusion formula we have:

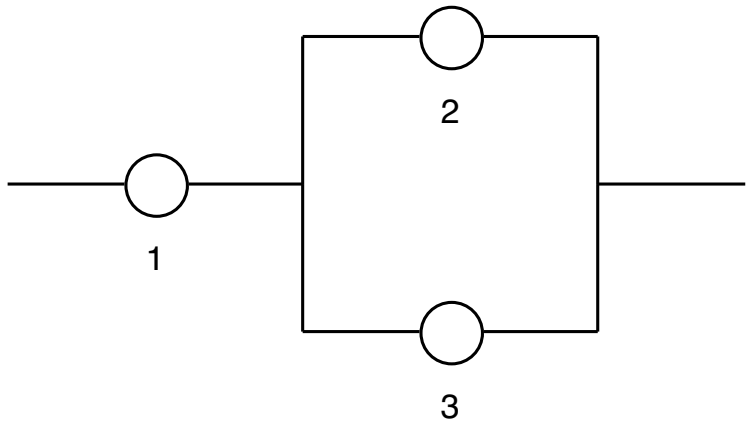
$$
\begin{aligned}
P(\phi = 1) &= P(\bigcup_{j=1}^{p} E_j) \\
&= P(E_1) + P(E_2) + \cdots + P(E_p) \\
&\quad - P(E_1 \cap E_2) - P(E_1 \cap E_3) - \cdots - P(E_{p-1} \cap E_p) \\
&\quad + P(E_1 \cap E_2 \cap E_3) + \cdots + P(E_{p-2} \cap E_{p-1} \cap E_p) \\
&\quad \cdots \\
&\quad + (-1)^{p-1} P(E_1 \cap \cdots \cap E_p).
\end{aligned}
$$

In the general case the number of terms in this formula before simplification is $2^p - 1$. Here $p = 12$, and the number of terms is $2^{12} - 1 = 4095$.

We observe that computing the reliability of this particular system using the inclusion-exclusion formula is very inefficient compared to using the simple formula derived in (e).



STK3405/4405 - Exam 2023, problem 2



Consider the binary monotone system (C, ϕ) shown above, where $C = \{1, 2, 3\}$. For $t \geq 0$ and $i \in C$ we let:

$$X_i(t) = \begin{cases} 1 & \text{if component } i \text{ is functioning at time } t \\ 0 & \text{otherwise} \end{cases}$$



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Moreover, let:

$$\boldsymbol{X}(t) = (X_1(t), X_2(t), X_3(t))$$

We assume that the component processes $\{X_1(t)\}, \{X_2(t)\}, \{X_3(t)\}$ are independent.

The lifetime of component i is denoted by T_i, and we assume that the survival function of T_i is $\bar{F}_i$, for $i \in C$.

Since the component processes are assumed to be independent, it follows that T_1, T_2, T_3 are independent as well, and we have:

$$P(T_i > t) = \bar{F}_i(t), \quad i \in C, \quad t \geq 0,$$



STK3405/4405 - Exam 2023, problem 2a

a) Show that for $t \geq 0$ we have:

$$P(\phi(\boldsymbol{X}(t)) = 1) = \bar{F}_1(t)\bar{F}_2(t) + \bar{F}_1(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t).$$

SOLUTION: We observe that the system consists of component 1 in series with the parallel connection of components 2 and 3. Thus, the structure function is:

$$\begin{aligned}
\phi(\boldsymbol{X}(t)) &= X_1(t)(X_2(t) \amalg X_3(t)) \\
&= X_1(t)(X_2(t) + X_3(t) - X_2(t)X_3(t)) \\
&= X_1(t)X_2(t) + X_1(t)X_3(t) - X_1(t)X_2(t)X_3(t)
\end{aligned}$$



STK3405/4405 - Exam 2023, problem 2a (cont.)

Furthermore, $X_i(t) = 1$ if and only if $T_i > t$, $i \in C$. Hence, we have:

$$\mathrm{E}[X_i(t)] = P(X_i(t) = 1) = P(T_i > t) = \bar{F}_i(t), \quad i \in C, \quad t \geq 0.$$

Hence, we have:

$$\begin{aligned} P(\phi(\boldsymbol{X}(t)) = 1) &= \mathrm{E}[\phi(\boldsymbol{X}(t))] \\ &= \mathrm{E}[X_1(t)X_2(t)] + \mathrm{E}[X_1(t)X_3(t)] - \mathrm{E}[X_1(t)X_2(t)X_3(t)] \\ &= \mathrm{E}[X_1(t)]\,\mathrm{E}[X_2(t)] + \mathrm{E}[X_1(t)]\,\mathrm{E}[X_3(t)] \\ &\quad - \mathrm{E}[X_1(t)]\,\mathrm{E}[X_2(t)]\,\mathrm{E}[X_3(t)] \\ &= \bar{F}_1(t)\bar{F}_2(t) + \bar{F}_1(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t), \end{aligned}$$

where the third equation follows since $\{X_1(t)\}, \{X_2(t)\}, \{X_3(t)\}$ are independent



STK3405/4405 - Exam 2023, problem 2b

The *time dependent* Birnbaum measure of *reliability importance* of component $i \in C$ at time $t \geq 0$ is defined as:

$$I_B^{(i)}(t) = P(\text{Component } i \text{ is critical for the system at time } t)$$

b) Show that for $t \geq 0$ we have:

$$I_B^{(1)}(t) = \bar{F}_2(t) + \bar{F}_3(t) - \bar{F}_2(t)\bar{F}_3(t),$$
$$I_B^{(2)}(t) = \bar{F}_1(t)(1 - \bar{F}_3(t)),$$
$$I_B^{(3)}(t) = \bar{F}_1(t)(1 - \bar{F}_2(t)).$$



STK3405/4405 - Exam 2023, problem 2b (cont.)

SOLUTION: We have:

$$\begin{aligned} I_B^{(i)}(t) &= P(\text{Component } i \text{ is critical for the system at time } t) \\ &= P(\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t)) = 1) \\ &= \mathrm{E}[\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t))] \\ &= \mathrm{E}[\phi(1_i, \boldsymbol{X}(t)] - \mathrm{E}[\phi(0_i, \boldsymbol{X}(t))] \end{aligned}$$

Since:

$$\mathrm{E}[\phi(\boldsymbol{X}(t)] = \bar{F}_1(t)\bar{F}_2(t) + \bar{F}_1(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t)$$

we get:

$$\begin{aligned} I_B^{(1)}(t) &= 1 \cdot \bar{F}_2(t) + 1 \cdot \bar{F}_3(t) - 1 \cdot \bar{F}_2(t)\bar{F}_3(t) \\ &\quad - 0 \cdot \bar{F}_2(t) - 0 \cdot \bar{F}_3(t) + 0 \cdot \bar{F}_2(t)\bar{F}_3(t) \\ &= \bar{F}_2(t) + \bar{F}_3(t) - \bar{F}_2(t)\bar{F}_3(t) \end{aligned}$$



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Using again that:

$$\mathrm{E}[\phi(\boldsymbol{X}(t)] = \bar{F}_1(t)\bar{F}_2(t) + \bar{F}_1(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t)$$

We also get:

$$\begin{aligned} I_B^{(2)}(t) &= 1 \cdot \bar{F}_1(t) + \bar{F}_1(t)\bar{F}_3(t) - 1 \cdot \bar{F}_1(t)\bar{F}_3(t) \\ &\quad - 0 \cdot \bar{F}_1(t) - \bar{F}_1(t)\bar{F}_3(t) + 0 \cdot \bar{F}_1(t)\bar{F}_3(t) \\ &= \bar{F}_1(t) - \bar{F}_1(t)\bar{F}_3(t) = \bar{F}_1(t)(1 - \bar{F}_3(t)) \end{aligned}$$

$$\begin{aligned} I_B^{(3)}(t) &= \bar{F}_1(t)\bar{F}_2(t) + 1 \cdot \bar{F}_1(t) - 1 \cdot \bar{F}_1(t)\bar{F}_2(t) \\ &\quad - \bar{F}_1(t)\bar{F}_2(t) - 0 \cdot \bar{F}_1(t) + 0 \cdot \bar{F}_1(t)\bar{F}_2(t) \\ &= \bar{F}_1(t) - \bar{F}_1(t)\bar{F}_2(t) = \bar{F}_1(t)(1 - \bar{F}_2(t)) \end{aligned}$$



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ALT. SOLUTION: We introduce $\boldsymbol{p} = (p_1, p_2, p_3)$, where $p_i = \bar{F}_i(t)$, $i \in C$. Then the reliability of the system can be written as:

$$P(\phi(\boldsymbol{X}(t)) = 1) = h(\boldsymbol{p}) = p_1 p_2 + p_1 p_3 - p_1 p_2 p_3$$

Since the component state processes are independent, we have that:

$$I_B^{(i)}(t) = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \quad i \in C.$$

Hence, we get:

$$I_B^{(1)}(t) = p_2 + p_3 - p_2 p_3 = \bar{F}_2(t) + \bar{F}_3(t) - \bar{F}_2(t)\bar{F}_3(t)$$

$$I_B^{(2)}(t) = p_1 - p_1 p_3 = \bar{F}_1(t)(1 - \bar{F}_3(t))$$

$$I_B^{(3)}(t) = p_1 - p_1 p_2 = \bar{F}_1(t)(1 - \bar{F}_2(t)).$$



STK3405/4405 - Exam 2023, problem 2c

The Barlow-Proschan measure of *reliability importance* of component $i \in C$ is defined as:

$$I_{B-P}^{(i)} = P(\text{Component } i \text{ fails at the same time as the system})$$

Assuming that T_1, T_2, T_3 are absolutely continuously distributed with densities f_1, f_2, f_3 respectively, it can be shown that we have:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) \cdot f_i(t) dt, \quad i \in C. \tag{1}$$

Finally, we assume that:

$$\bar{F}_i(t) = e^{-\lambda_i t}, \quad i \in C, \quad t \geq 0.$$

where $\lambda_i > 0$, for all $i \in C$.



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In the calculations you may use the following formula:

$$\int_0^\infty e^{-at}dt = \frac{1}{a}, \quad \text{for all } a > 0. \tag{2}$$

c) By using (1), (2) and the results from (b) show that:

$$I_{B-P}^{(1)} = \frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_3} - \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3},$$

$$I_{B-P}^{(2)} = \frac{\lambda_2}{\lambda_1+\lambda_2} - \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3},$$

$$I_{B-P}^{(3)} = \frac{\lambda_3}{\lambda_1+\lambda_3} - \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}.$$

Show that $I_{B-P}^{(1)} + I_{B-P}^{(2)} + I_{B-P}^{(3)} = 1$, and explain briefly why this must hold.



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SOLUTION: We first note that since $\bar{F}_i(t) = e^{-\lambda_i t}$, $i \in C$, $t \geq 0$, the densities of the lifetimes T_1, T_2, T_3 are given by:

$$f_i(t) = -\frac{d}{dt}\bar{F}_i(t) = \lambda_i e^{-\lambda_i t} = \lambda_i \bar{F}_i(t), \quad i \in C, \quad t \geq 0.$$

Thus, $T_i \sim Exp(\lambda_i)$, $i \in C$. By combining (b) and (1) we get:

$$\begin{aligned}
I^{(1)}_{B-P} &= \int_0^\infty [\bar{F}_2(t) + \bar{F}_3(t) - \bar{F}_2(t)\bar{F}_3(t)] f_1(t) dt \\
&= \lambda_1 \int_0^\infty [\bar{F}_1(t)\bar{F}_2(t) + \bar{F}_1(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t)] dt \\
&= \lambda_1 \left[\int_0^\infty e^{-(\lambda_1+\lambda_2)t} dt + \int_0^\infty e^{-(\lambda_1+\lambda_3)t} dt - \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_3)t} dt \right] \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_3} - \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3},
\end{aligned}$$

where the last equality follows by applying the integral formula (2).



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$$
\begin{aligned}
I^{(2)}_{B-P} &= \int_0^\infty [\bar{F}_1(t) - \bar{F}_1(t)\bar{F}_3(t)] f_2(t) dt \\
&= \lambda_2 \int_0^\infty [\bar{F}_1(t)\bar{F}_2(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t)] dt \\
&= \lambda_2 \left[\int_0^\infty e^{-(\lambda_1+\lambda_2)t} dt - \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_3)t} dt \right] \\
&= \frac{\lambda_2}{\lambda_1+\lambda_2} - \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}
\end{aligned}
$$

where the last equality follows by applying the integral formula (2).



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$$
\begin{aligned}
I_{B-P}^{(3)} &= \int_0^\infty [\bar{F}_1(t) - \bar{F}_1(t)\bar{F}_2(t)] f_3(t) dt \\
&= \lambda_3 \int_0^\infty [\bar{F}_1(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t)] dt \\
&= \lambda_3 \left[\int_0^\infty e^{-(\lambda_1+\lambda_3)t} dt - \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_3)t} dt \right] \\
&= \frac{\lambda_3}{\lambda_1+\lambda_3} - \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}
\end{aligned}
$$

where the last equality follows by applying the integral formula (2).



STK3405/4405 - Exam 2023, problem 2c (cont.)

We observe that:

$$I_{B-P}^{(1)} + I_{B-P}^{(2)} + I_{B-P}^{(3)}$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_3} - \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}$$

$$+ \frac{\lambda_2}{\lambda_1+\lambda_2} - \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} + \frac{\lambda_3}{\lambda_1+\lambda_3} - \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}$$

$$= \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_3} - \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2+\lambda_3} = 1+1-1=1$$



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We let S denote the lifetime of the system. Since the system can only fail as a result of a component failure, we have:

$$P(\bigcup_{i=1}^{3}\{T_i = S\}) = 1.$$

Since T_1, T_2, T_3 are independent and absolutely continuously distributed, the probability that two or more components fail at the same time is zero.

This implies that $P(\{T_j = S\} \cap \{T_i = S\}) = 0$ for $i \neq j$, and that $P(\{T_1 = S\} \cap \{T_2 = S\} \cap \{T_3 = S\}) = 0$.

Hence, we get:

$$1 = P(\bigcup_{i=1}^{n}\{T_i = S\}) = \sum_{i=1}^{n} P(T_i = S) = \sum_{i=1}^{n} I_{B-P}^{(i)},$$



STK3405/4405 - Exam 2023, problem 2d

d) Calculate $I^{(1)}_{B-P}$, $I^{(2)}_{B-P}$ and $I^{(3)}_{B-P}$ assuming that $\lambda_2 = \lambda_1$ and $\lambda_3 = 3\lambda_1$. Comment the resulting ranking of the components with respect to their importances.

SOLUTION: Assuming that $\lambda_2 = \lambda_1$ and $\lambda_3 = 3\lambda_1$, we get:

$$I^{(1)}_{B-P} = \frac{\lambda_1}{\lambda_1 + \lambda_1} + \frac{\lambda_1}{\lambda_1 + 3\lambda_1} - \frac{\lambda_1}{\lambda_1 + \lambda_1 + 3\lambda_1}$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{5} = \frac{10 + 5 - 4}{20} = \frac{11}{20}.$$



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Furthermore:

$$I^{(2)}_{B-P} = \frac{\lambda_1}{\lambda_1 + \lambda_1} - \frac{\lambda_1}{\lambda_1 + \lambda_1 + 3\lambda_1}$$
$$= \frac{1}{2} - \frac{1}{5} = \frac{10-4}{20} = \frac{6}{20}.$$

$$I^{(3)}_{B-P} = \frac{\lambda_1}{\lambda_1 + 3\lambda_1} - \frac{3\lambda_1}{\lambda_1 + \lambda_1 + 3\lambda_1}$$
$$= \frac{3}{4} - \frac{3}{5} = \frac{15-12}{20} = \frac{3}{20}.$$



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From this we conclude that:

$$I_{B-P}^{(1)} > I_{B-P}^{(2)} > I_{B-P}^{(3)}$$

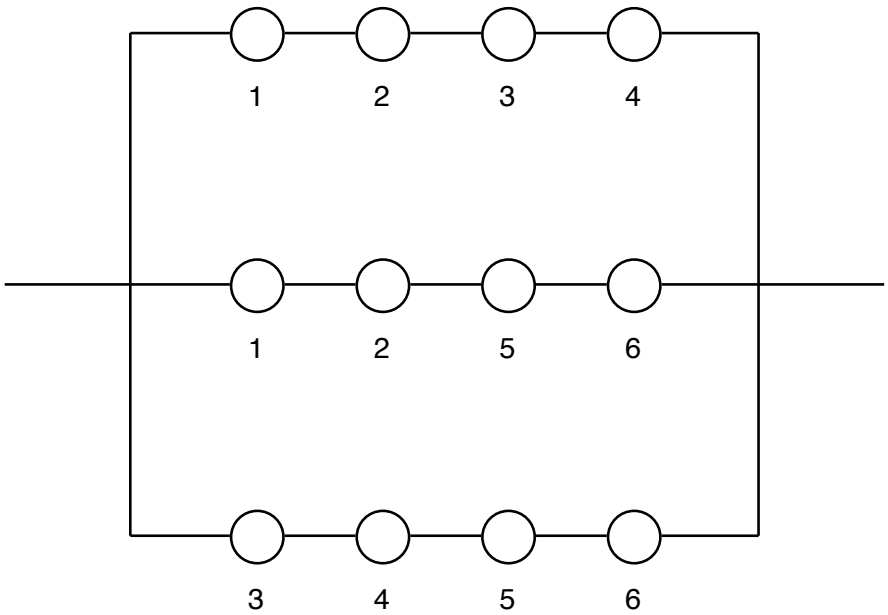
That is, component 1 is the most important component, component 2 is the second most important component, while component 3 is the least important component.

That component 1 is the most important component is reasonable since this component is in series with the rest of the system.

That component 2 is more important than component 3 is also reasonable since in a parallel connection the best component, i.e., the one with the lowest failure rate, is the most important component



STK3405/4405 - Exam 2023, problem 1



Consider the binary monotone system (C, ϕ) shown above with component set $C = \{1, \ldots, 6\}$. Let $\boldsymbol{X} = (X_1, \ldots, X_6)$ denote the vector of component state variables, and assume throughout this problem that $X_1, \ldots, X_6$ are stochastically independent, and that $P(X_i = 1) = p$, for all $i \in C$.



STK3405/4405 - Exam 2023, problem 1a

a) Find the minimal path sets (3 sets) and the minimal cut sets (12 sets) of the system. Is the system *coherent*?

SOLUTION: Minimal path sets:

$$P_1 = \{1, 2, 3, 4\},\ P_2 = \{1, 2, 5, 6\},\ P_3 = \{3, 4, 5, 6\}$$

Minimal cut sets:

$$K_1 = \{1, 3\},\ K_2 = \{1, 4\},\ K_3 = \{2, 3\},\ K_4 = \{2, 4\}$$
$$K_5 = \{1, 5\},\ K_6 = \{1, 6\},\ K_7 = \{2, 5\},\ K_8 = \{2, 6\}$$
$$K_9 = \{3, 5\},\ K_{10} = \{3, 6\},\ K_{11} = \{4, 5\},\ K_{12} = \{4, 6\}.$$

Since $P_1 \cup P_2 \cup P_3 = C$, all components are contained in at least one minimal path set. Hence, it follows that the system is *coherent*.

Alternatively, since $K_1 \cup \ldots \cup K_{12} = C$, all components are contained in at least one minimal cut set. Hence, it follows that the system is *coherent*



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We introduce the modules (A_1, χ_1), (A_2, χ_2) and (A_3, χ_3), where $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$ and $A_3 = \{5, 6\}$, and where:

$$\chi_1(X_1, X_2) = X_1 \cdot X_2$$
$$\chi_2(X_3, X_4) = X_3 \cdot X_4$$
$$\chi_3(X_5, X_6) = X_5 \cdot X_6$$

b) Explain why the structure function of the system is given by:

$$\phi(\boldsymbol{X}) = \mathsf{I}(\chi_1(X_1, X_2) + \chi_2(X_3, X_4) + \chi_3(X_5, X_6) \geq 2)$$

where $\mathsf{I}(\cdot)$ denotes the indicator function.



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SOLUTION: We observe that the system can be interpreted as a 2-out-of-3 system, but where the components are the modules (A_1, χ_1), (A_2, χ_2) and (A_3, χ_3).

The structure function of a 2-out-of-3 system can be written as the indicator function of the event that the sum of the component states is greater than or equal to 2.

Hence, it follows that:

$$\phi(\boldsymbol{X}) = \mathsf{I}(\chi_1(X_1, X_2) + \chi_2(X_3, X_4) + \chi_3(X_5, X_6) \geq 2)$$

as claimed.



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We then let:

$$S = S(\boldsymbol{X}) = \chi_1(X_1, X_2) + \chi_2(X_3, X_4) + \chi_3(X_5, X_6)$$

c) Explain briefly why:

$$P(S = s) = \binom{3}{s}(p^2)^s(1 - p^2)^{3-s}, \quad s = 0, 1, 2, 3.$$



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SOLUTION: We observe that S is the sum three independent and identically distributed binary variables.

Hence, S is binomially distributed. It is easy to see that $P(\chi_i = 1) = p^2$ for $i = 1, 2, 3$. Hence, it follows that $S \sim Bin(3, p^2)$.

Thus, we have:

$$P(S = s) = \binom{3}{s}(p^2)^s(1 - p^2)^{3-s}, \quad s = 0, 1, 2, 3.$$



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d) Explain briefly why:

$$P(\phi = 1) = P(S = 2) + P(S = 3),$$

and indicate how this can be used to show that:

$$P(\phi = 1) = 3p^4(1 - p^2) + p^6 = 3p^4 - 2p^6$$

SOLUTION: We have that:

$$\begin{aligned} P(\phi = 1) = P(S \geq 2) &= P(S = 2) + P(S = 3) \\ &= \binom{3}{2}(p^2)^2(1 - p^2) + \binom{3}{3}(p^2)^3 \\ &= 3p^4(1 - p^2) + p^6 = 3p^4 - 2p^6 \end{aligned}$$

