STK3405 – Lecture 25

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STK3405/4405 - Exam 2024, problem 1a

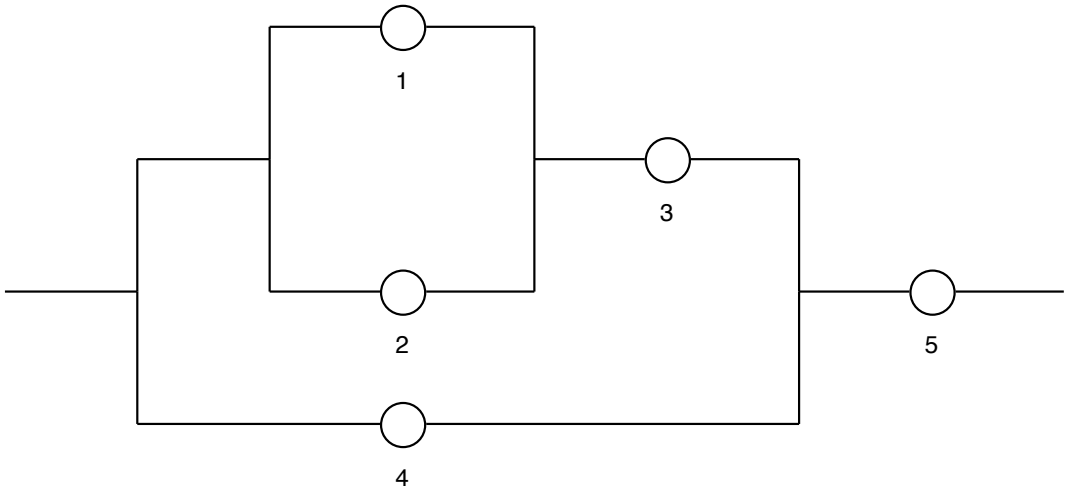


Figure: Block diagram the binary monotone system (C, ϕ).

The component set of the system is $C = \{1, \ldots, 5\}$. $\boldsymbol{X} = (X_1, \ldots, X_5)$ is the vector of component state variables.

Assume throughout this problem that $X_1, \ldots, X_5$ are independent, and that $\Pr(X_i = 1) = p_i$, for all $i \in C$. We also let $\boldsymbol{p} = (p_1, \ldots, p_5)$.



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a) Find the minimal path sets (3 sets) and the minimal cut sets (3 sets) of the system. Is the system coherent?

SOLUTION: Minimal path sets:

$$P_1 = \{1, 3, 5\},\ P_2 = \{2, 3, 5\},\ P_3 = \{4, 5\}.$$

Minimal cut sets:

$$K_1 = \{1, 2, 4\},\ K_2 = \{3, 4\},\ K_3 = \{5\}$$

Since $P_1 \cup P_2 \cup P_3 = C$, all components are contained in at least one minimal path set. Hence, it follows that the system is coherent.

Alternatively, since $K_1 \cup K_2 \cup K_3 = C$, all components are contained in at least one minimal cut set. Hence, it follows that the system is coherent.



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b) Explain why the structure function of the system is given by:

$$\phi(\boldsymbol{X}) = ([(X_1 \amalg X_2) \cdot X_3] \amalg X_4) \cdot X_5,$$

and use this to show that that the reliability function of the system is given by:

$$\begin{aligned} h(\boldsymbol{p}) &= ([(p_1 \amalg p_2) \cdot p_3] \amalg p_4) \cdot p_5 \\ &= [(p_1 + p_2 - p_1 p_2) p_3 (1 - p_4) + p_4] \, p_5 \end{aligned}$$



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SOLUTION: The structure function can be derived by using the fact that the system is an *sp-system* (series-parallel system). I more detail, we see that the structure function is obtained using the following reductions:

- Components 1 and 2 are in parallel, and can be reduced to a single component, $2'$, where $X_{2'} = (X_1 \amalg X_2)$.
- Components $2'$ and 3 are in series, and can be reduced to a single component, $3'$, where $X_{3'} = (X_2' \cdot X_3)$.
- Components $3'$ and 4 are in parallel, and can be reduced to a single component, $4'$, where $X_{4'} = (X_2' \amalg X_3)$.
- Components $4'$ and 5 are in series, and can be reduced to a single component, $5'$, where $X_{5'} = (X_4' \cdot X_5)$.

By inserting the expressions for $X_{2'}$, $X_{3'}$, $X_{4'}$ and $X_{5'}$, we get the desired structure function:

$$\phi(\boldsymbol{X}) = ([(X_1 \amalg X_2) \cdot X_3] \amalg X_4) \cdot X_5,$$



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Since we have assumed that the component state variables, $X_1, \ldots, X_5$ are stochastically independent, it follows that:

$$
\begin{aligned}
h(\boldsymbol{p}) &= \mathrm{E}[\phi(\boldsymbol{X})] \\
&= \mathrm{E}[[(X_1 \amalg X_2) \cdot X_3] \amalg X_4) \cdot X_5] \\
&= [[(\mathrm{E}[X_1] \amalg \mathrm{E}[X_2]) \cdot \mathrm{E}[X_3]] \amalg \mathrm{E}[X_4])] \cdot \mathrm{E}[X_5] \\
&= ([(p_1 \amalg p_2) \cdot p_3] \amalg p_4) \cdot p_5 \\
&= ([(p_1 + p_2 - p_1 p_2) p_3] \amalg p_4) \cdot p_5 \\
&= [(p_1 + p_2 - p_1 p_2) p_3 + p_4 - (p_1 + p_2 - p_1 p_2) p_3 p_4] \, p_5 \\
&= [(p_1 + p_2 - p_1 p_2) p_3 (1 - p_4) + p_4] \, p_5
\end{aligned}
$$



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The Birnbaum measure of reliability importance of component i, denoted $I_B^{(i)}$, is defined by:

$$\begin{aligned} I_B^{(i)} &= \Pr(\text{Component } i \text{ is critical for the system}) \\ &= \Pr(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1), \quad i = 1, \ldots, 5. \end{aligned}$$

c) Show that:

$$\begin{aligned} I_B^{(1)} &= (1 - p_2)p_3(1 - p_4)p_5, \\ I_B^{(2)} &= (1 - p_1)p_3(1 - p_4)p_5, \\ I_B^{(3)} &= (p_1 + p_2 - p_1p_2)(1 - p_4)p_5, \\ I_B^{(4)} &= [1 - (p_1 + p_2 - p_1p_2)p_3]\, p_5, \\ I_B^{(5)} &= [(p_1 + p_2 - p_1p_2)p_3(1 - p_4) + p_4] \end{aligned}$$



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SOLUTION: Since $X_1, \ldots, X_5$ are independent, we have:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \quad i = 1, \ldots, 5.$$

Hence, since $h(\boldsymbol{p}) = [(p_1 + p_2 - p_1 p_2)p_3(1 - p_4) + p_4]\, p_5$, we get:

$$I_B^{(1)} = (1 - p_2)p_3(1 - p_4)p_5,$$

$$I_B^{(2)} = (1 - p_1)p_3(1 - p_4)p_5,$$

$$I_B^{(3)} = (p_1 + p_2 - p_1 p_2)(1 - p_4)p_5,$$

$$I_B^{(4)} = [1 - (p_1 + p_2 - p_1 p_2)p_3]\, p_5,$$

$$I_B^{(5)} = [(p_1 + p_2 - p_1 p_2)p_3(1 - p_4) + p_4]$$



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In the remaining part of this problem we assume that $p_1 = \cdots = p_5 = p$, where $0 < p < 1$.

d) Show that:

$$I_B^{(1)} = p^2(1-p)^2,$$

$$I_B^{(2)} = p^2(1-p)^2,$$

$$I_B^{(3)} = p^2(1-p)(2-p),$$

$$I_B^{(4)} = \left[1 - p^2(2-p)\right] p,$$

$$I_B^{(5)} = \left[p^2(1-p)(2-p) + p\right]$$



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SOLUTION: By inserting $p_1 = \cdots = p_5 = p$, we get:

$$I_B^{(1)} = (1-p)p(1-p)p = p^2(1-p)^2,$$

$$I_B^{(2)} = (1-p)p(1-p)p = p^2(1-p)^2,$$

$$I_B^{(3)} = (2p-p^2)(1-p)p = p^2(1-p)(2-p),$$

$$I_B^{(4)} = \left[1-(2p-p^2)p\right]p = \left[1-p^2(2-p)\right]p,$$

$$I_B^{(5)} = \left[(2p-p^2)p(1-p)+p\right] = \left[p^2(1-p)(2-p)+p\right]$$



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e) Show that: $I_B^{(1)} = I_B^{(2)} < I_B^{(3)} < I_B^{(5)}$

SOLUTION: We recall that:

$$
\begin{aligned}
I_B^{(1)} &= I_B^{(2)} = p^2(1-p)^2, \\
I_B^{(3)} &= p^2(1-p)(2-p), \\
I_B^{(4)} &= \left[1 - p^2(2-p)\right] p, \\
I_B^{(5)} &= \left[p^2(1-p)(2-p) + p\right]
\end{aligned}
$$

Since $I_B^{(1)} = I_B^{(2)} = p^2(1-p)^2$, the first equality is trivial.

Since $(1-p) < (2-p)$, it follows that $p^2(1-p)^2 < p^2(1-p)(2-p)$. Thus, $I_B^{(2)} < I_B^{(3)}$.

Finally, we observe that $I_B^{(5)} = I_B^{(3)} + p$. Thus, $I_B^{(3)} < I_B^{(5)}$.



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f) Show that:

$$I_B^{(3)} < I_B^{(4)} < I_B^{(5)}$$

[Hint: Explain why $p(2-p) < 1$ for all $p < 1$]

SOLUTION: We observe that $p(2-p)$ is a parabola in p with a maximum value of 1, which corresponds to $p = 1$. Thus, it follows that:

$$p(2-p) < 1, \quad \text{for all } p < 1.$$



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This implies that:

$$I_B^{(3)} = p^2(1-p)(2-p) < p(1-p)$$

$$I_B^{(4)} = \left[1 - p^2(2-p)\right] p > [1-p]\, p$$

Thus, it follows that $I_B^{(3)} < I_B^{(4)}$.

In order to show that $I_B^{(4)} < I_B^{(5)}$, we note that $p^2(2-p) > 0$ and that $p^2(1-p)(2-p) > 0$. Hence, it follows that

$$I_B^{(4)} = \left[1 - p^2(2-p)\right] p < [1-0]\, p = p$$

$$< \left[p^2(1-p)(2-p) + p\right] = I_B^{(5)}$$



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A binary vector $(\cdot_i, \boldsymbol{x})$ is said to be a *critical vector* for component i if:

$$\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x}) = 1.$$

g) Show that the number of critical vectors for component 5 is 11.

SOLUTION: We recall that the Birnbaum measure of structural importance is defined as:

$$J_B^{(i)} = \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})]$$

Hence, the number of critical vectors for component i is $2^{n-1} J_B^{(i)}$.



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We also recall that $J_B^{(i)} = I_B^{(i)}$ whenever $p_1 = \cdots = p_n = \frac{1}{2}$.

Since we know that:

$$I_B^{(5)} = \left[p^2(1-p)(2-p) + p\right],$$

we get:

$$J_B^{(5)} = \left[(\tfrac{1}{2})^2(1-\tfrac{1}{2})(2-\tfrac{1}{2}) + \tfrac{1}{2}\right]$$

$$= \left[\tfrac{1}{4} \cdot \tfrac{1}{2} \cdot \tfrac{3}{2} + \tfrac{1}{2}\right] = \tfrac{3}{16} + \tfrac{8}{16} = \tfrac{11}{16} = \frac{1}{2^{5-1}} \cdot 11$$

Hence, the number of critical vectors for component 5 is 11.



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ALT. SOLUTION: Since component 5 is in series with the rest of the system, it follows that:

$$\phi(0_5, \boldsymbol{x}) = 0$$

Hence, component 5 is critical if and only if:

$$\phi(1_5, \boldsymbol{x}) - \phi(0_5, \boldsymbol{x}) = \phi(1_5, \boldsymbol{x}) = 1.$$

Equivalently, component 5 is not critical if and only if:

$$\phi(1_5, \boldsymbol{x}) = ([(x_1 \amalg x_2) \cdot x_3] \amalg x_4) = 0$$



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The total number of binary vectors $(1_5, \boldsymbol{x})$ is $2^{5-1} = 2^4 = 16$. Our goal is to identify the critical vectors for component 5 among these 16 vectors.

We start out by noting that if component 5 is functioning, component 4 is in parallel with the branch consisting of components 1, 2 and 3.

This implies that we have:

$$\phi(1_4, 1_5, \boldsymbol{x}) = 1 \quad \text{for all } (1_4, 1_5, \boldsymbol{x})$$

The total number of binary vectors of form $(1_4, 1_5, \boldsymbol{x})$ is $2^{5-2} = 2^3 = 8$.

It follows that for all these 8 vectors component 5 is critical.



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It remains to consider all binary vectors of form $(0_4, 1_5, \boldsymbol{x})$. The total number of such vectors is also $2^{5-2} = 2^3 = 8$. It is easy to verify that:

$$\begin{aligned}
\phi(0,0,0,0,1) &= 0 \\
\phi(1,0,0,0,1) &= 0 \\
\phi(0,1,0,0,1) &= 0 \\
\phi(1,1,0,0,1) &= 0 \\
\phi(0,0,1,0,1) &= 0 \\
\phi(1,0,1,0,1) &= 1 \\
\phi(0,1,1,0,1) &= 1 \\
\phi(1,1,1,0,1) &= 1
\end{aligned}$$

Thus, for the first 5 vectors component 5 is not critical, while for the last 3 vectors component 5 is critical.

Thus, for 5 out of the 16 vectors component 5 is not critical. For remaining $8 + 3 = 11$ vectors component 5 is critical.



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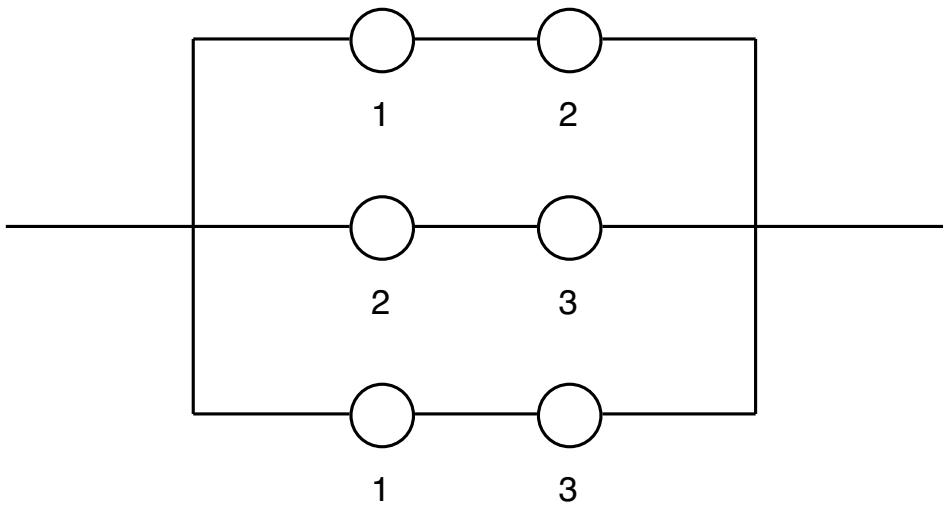


Figure: Block diagram the binary monotone system (C, ϕ).

The components are *repairable*, and the component state processes are independent.

The component state vector at time $t \geq 0$ is $\boldsymbol{X}(t) = (X_1(t), X_2(t), X_3(t))$.



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The stationary component availabilities, denoted A_1, A_2, A_3, are defined by:

$$A_i = \lim_{t\to\infty} \Pr(X_i(t) = 1) = \lim_{t\to\infty} \mathrm{E}[X_i(t)], \qquad i = 1, 2, 3,$$

The stationary system availability, denoted A_ϕ, is defined by:

$$A_\phi = \lim_{t\to\infty} \Pr(\phi(\boldsymbol{X}(t)) = 1) = \lim_{t\to\infty} \mathrm{E}[\phi(\boldsymbol{X}(t))].$$

a) Show that:

$$A_\phi = A_1A_2 + A_1A_3 + A_2A_3 - 2A_1A_2A_3.$$



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SOLUTION: We observe that (C, ϕ) is a 2-out-of-3 system. Thus, the structure function, ϕ, is given by:

$$\begin{aligned}\phi(\boldsymbol{X}(t)) &= (X_1(t)X_2(t)) \amalg (X_1(t)X_3(t)) \amalg (X_2(t)X_3(t)) \\ &= 1 - (1 - X_1(t)X_2(t)) \cdot (1 - X_1(t)X_3(t)) \cdot (1 - X_2(t)X_3(t)) \\ &= X_1(t)X_2(t) + X_1(t)X_3(t) + X_2(t)X_3(t) - 2X_1(t)X_2(t)X_3(t)\end{aligned}$$

By taking the expectation on both sides, using that the component state processes are independent, and letting t go to infinity, we get:

$$A_\phi = A_1A_2 + A_1A_3 + A_2A_3 - 2A_1A_2A_3.$$



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The stationary Birnbaum importance measure of component i is denoted $I_B^{(i)}$, and is defined by:

$$I_B^{(i)} = \lim_{t\to\infty} \Pr(\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t)) = 1)$$

$$= \lim_{t\to\infty} \mathrm{E}[\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t))], \quad i = 1, 2, 3.$$

b) Show that:

$$I_B^{(1)} = A_2 + A_3 - 2A_2A_3,$$
$$I_B^{(2)} = A_1 + A_3 - 2A_1A_3,$$
$$I_B^{(3)} = A_1 + A_2 - 2A_1A_2$$



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SOLUTION: Using the expression for the structure function we get:

$$\phi(1_1, \boldsymbol{X}(t)) - \phi(0_1, \boldsymbol{X}(t)) = X_2(t) + X_3(t) - 2X_2(t)X_3(t)$$

$$\phi(1_2, \boldsymbol{X}(t)) - \phi(0_2, \boldsymbol{X}(t)) = X_1(t) + X_3(t) - 2X_1(t)X_3(t)$$

$$\phi(1_3, \boldsymbol{X}(t)) - \phi(0_3, \boldsymbol{X}(t)) = X_1(t) + X_2(t) - 2X_1(t)X_2(t)$$

By taking the expectation on both sides, using that the component state processes are independent, and letting t go to infinity, we get:

$$I_B^{(1)} = A_2 + A_3 - 2A_2A_3,$$
$$I_B^{(2)} = A_1 + A_3 - 2A_1A_3,$$
$$I_B^{(3)} = A_1 + A_2 - 2A_1A_2$$



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Let μ_i and ν_i denote respectively the expected lifetimes and the expected repair times of component i, $i = 1, 2, 3$. Under suitable conditions, which we assume hold true here, it can be shown that:

$$A_i = \frac{\mu_i}{\mu_i + \nu_i}, \qquad i = 1, 2, 3,$$

c) In the remaining part of this problem we assume that $\mu_i = 3\nu_i$, $i = 1, 2, 3$. Show that in this case

$$I_B^{(1)} = I_B^{(2)} = I_B^{(3)} = \frac{3}{8}$$



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SOLUTION: By the assumption that $\mu_i = 3\nu_i$, $i = 1, 2, 3$, we get:

$$A_i = \frac{\mu_i}{\mu_i + \nu_i} = \frac{3\nu_i}{3\nu_i + \nu_i} = \frac{3}{4}, \quad i = 1, 2, 3.$$

We then insert these availabilities into the expressions for $I_B^{(1)}$, $I_B^{(2)}$ and $I_B^{(3)}$, and get:

$$I_B^{(i)} = \frac{3}{4} + \frac{3}{4} - 2 \cdot \frac{9}{16} = \frac{6 + 6 - 9}{8} = \frac{3}{8}, \quad i = 1, 2, 3.$$



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The Barlow-Proschan importance measure of component i, denoted $I_{B-P}^{(i)}$, is defined as the asymptotic fraction of system failures caused by component i, $i = 1, 2, 3$.

It can be shown that this is given by:

$$I_{B-P}^{(i)} = \frac{I_B^{(i)}/(\nu_i + \mu_i)}{\sum_{j=1}^{n} I_B^{(j)}/(\nu_j + \mu_j)}, \quad i = 1, 2, 3. \tag{1}$$

d) Assume that $\nu_1 = 1$, $\nu_2 = 2$ and $\nu_3 = 3$. Show that in this case we have:

$$I_{B-P}^{(1)} = \frac{6}{11},$$

$$I_{B-P}^{(2)} = \frac{3}{11},$$

$$I_{B-P}^{(3)} = \frac{2}{11}$$



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SOLUTION: We note that since we in this case have $I_B^{(1)} = I_B^{(2)} = I_B^{(3)}$, the expression for the importance measure $I_{B-P}^{(i)}$ can be simplified to:

$$I_{B-P}^{(i)} = \frac{(\nu_i + \mu_i)^{-1}}{\sum_{j=1}^{3}(\nu_j + \mu_j)^{-1}}$$

Moreover, since $\nu_1 = 1$, $\nu_2 = 2$ and $\nu_3 = 3$, and $\mu_i = 3\nu_i$, $i = 1, 2, 3$, we get:

$$\nu_1 + \mu_1 = 1 + 3 = 4$$

$$\nu_2 + \mu_2 = 2 + 6 = 8$$

$$\nu_3 + \mu_3 = 3 + 9 = 12$$

and thus:

$$\sum_{j=1}^{3}(\nu_j + \mu_j)^{-1} = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} = \frac{6+3+2}{24} = \frac{11}{24}$$



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Hence, it follows that:

$$I_{B-P}^{(1)} = \frac{\frac{6}{24}}{\frac{11}{24}} = \frac{6}{11},$$

$$I_{B-P}^{(2)} = \frac{\frac{3}{24}}{\frac{11}{24}} = \frac{3}{11},$$

$$I_{B-P}^{(3)} = \frac{\frac{2}{24}}{\frac{11}{24}} = \frac{2}{11}$$



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e) Give a brief comment to the results in (d). In particular, explain why we get a different ranking of the components when using the Barlow-Proschan importance measure compared to the Birnbaum importance measure.

SOLUTION: While the stationary Birnbaum importance measure depends only on the component availabilities, the Barlow-Proschan importance measure depends also on how often the components fail.

The expected life cycle of a component is the sum of the expected lifetime and the expected repair time.

A component with a short expected life cycle will fail more frequently than a component with a long expected life cycle, and hence potentially also cause more system failures.



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In this case all the components have the same availability, i.e., $\frac{3}{4}$. The expected life cycles, however, are different.

- Component 1 has the shortest expected life cycle, i.e., 4, and hence, according to the Barlow-Proschan measure, is the most important component.
- Component 2 has the second shortest expected life cycle, i.e., 8, and hence, according to the Barlow-Proschan measure, is the second most important component.
- Component 3 has the longest expected life cycle, i.e., 12, and hence, according to the Barlow-Proschan measure, is the least important component.

