STK3405 – Lecture 3

A. B. Huseby

Department of Mathematics
University of Oslo, Norway



Exercise 2.1

What is the reliability function of a series system of order n where the component states are assumed to be independent?

SOLUTION: A series system of order n is a binary monotone system (C, ϕ) which functions if and only if all the n components function. Thus, the structure function of the system is:

$$\phi(\boldsymbol{X}) = \prod_{i=1}^{n} X_i.$$

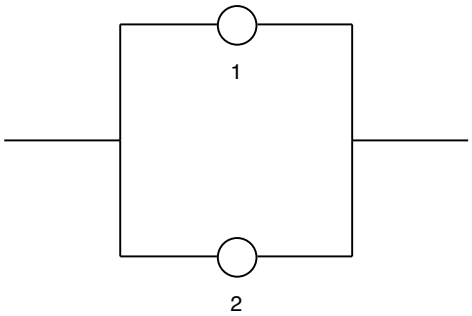
Assuming that the component state variables $X_1, \ldots, X_n$ are independent with reliabilities $p_1, \ldots, p_n$ respectively, we get that:

$$h(\boldsymbol{p}) = P(\phi(\boldsymbol{X}) = 1) = E[\phi(\boldsymbol{X})] = E[\prod_{i=1}^{n} X_i]$$

$$= \prod_{i=1}^{n} E[X_i] \text{ (using the independence) } = \prod_{i=1}^{n} p_i$$



Exercise 2.2



Consider the parallel structure of order 2. Assume that the component states are independent. The reliability function of this system is:

$$h(\boldsymbol{p}) = p_1 \amalg p_2 = 1 - (1 - p_1)(1 - p_2) = p_1 + p_2 - p_1 p_2.$$

a) Assume that $p_1 = P(X_1 = 1) = 0.4$ and $p_2 = P(X_2 = 1) = 0.6$. Compute the reliability of the system?

SOLUTION: $h(0.4, 0.6) = 0.4 + 0.6 - 0.4 \cdot 0.6 = 0.76$



Exercise 2.2 (cont.)

b) What is the system reliability if $p_1 = 0.9$ and $p_2 = 0.1$?

SOLUTION: $h(0.9, 0.1) = 0.9 + 0.1 - 0.9 \cdot 0.1 = 0.91$

c) Compare the results from (a) and (b). Note that $p_1 + p_2 = 1$ in both cases, but the system reliabilities are different. Can you explain why?

SOLUTION: For a parallel system it is better to have *one really good* component and *one really bad* component, than to have two components with reliabilities close 0.5.



Exercise 2.2 (cont.)

Assume more generally that $P(X_1 = 1) = p$ and $P(X_2 = 1) = 1 - p$. This implies that:

$$h(\boldsymbol{p}) = p + (1 - p) - p(1 - p) = 1 - p(1 - p).$$

This is parabola with minimum at $p = 0.5$. Thus, the system reliability of the parallel system is smallest when the components have the same reliability.

For a series system of two components with $P(X_1 = 1) = p$ and $P(X_2 = 1) = 1 - p$ we have:

$$h(\boldsymbol{p}) = p(1 - p).$$

This is parabola with maximum at $p = 0.5$. Thus, the system reliability of the series system is largest when the components have the same reliability.



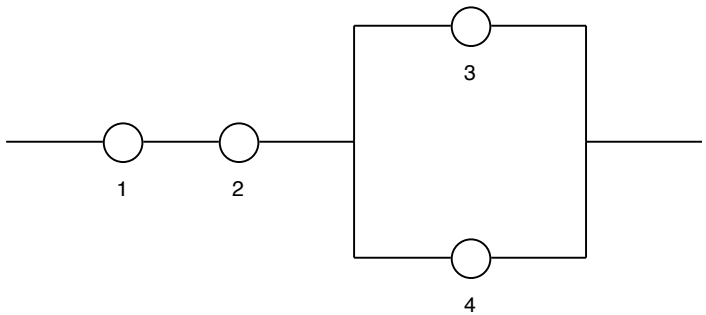
Exercise 2.3

Consider a binary monotone system (C, ϕ), where the component set is $C = \{1, \ldots, 4\}$ and where ϕ is given by:

$$\phi(\boldsymbol{x}) = x_1 \cdot x_2 \cdot (x_3 \amalg x_4).$$

a) Draw a reliability block diagram of this system.

SOLUTION:





Exercise 2.3 (cont.)

b) Assume again that ϕ is given by:

$$\phi(\boldsymbol{x}) = x_1 \cdot x_2 \cdot (x_3 \amalg x_4).$$

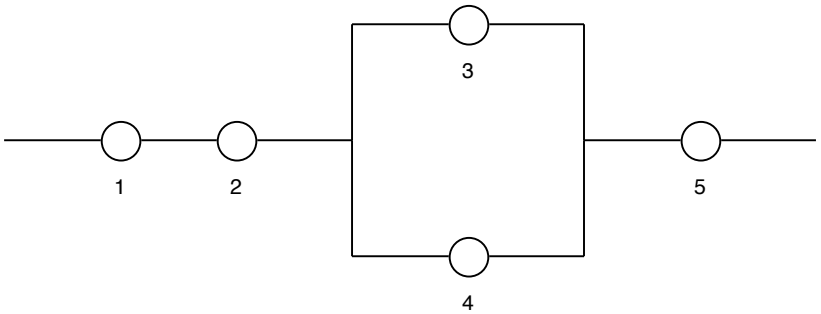
and that X_1, X_2, X_3, X_4 are independent with reliabilities p_1, p_2, p_3, p_4 respectively. What is the corresponding reliability function?

SOLUTION:

$$\begin{aligned} h(\boldsymbol{p}) &= E[\phi(\boldsymbol{X})] = E[X_1 \cdot X_2 \cdot (X_3 \amalg X_4)] \\ &= E[X_1] \cdot E[X_2] \cdot (E[X_3] \amalg E[X_4]) \quad \text{using independence} \\ &= p_1 \cdot p_2 \cdot (p_3 \amalg p_4) \end{aligned}$$



Exercise 2.4



a) What is the structure function of this system?

SOLUTION: The structure function of this system is:

$$\phi(\boldsymbol{x}) = x_1 \cdot x_2 \cdot (x_3 \amalg x_4) \cdot x_5.$$



Exercise 2.4 (cont.)

b) Assume again that ϕ is given by:

$$\phi(\boldsymbol{x}) = x_1 \cdot x_2 \cdot (x_3 \amalg x_4) \cdot x_5.$$

and that X_1, X_2, X_3, X_4, X_5 are independent with reliabilities p_1, p_2, p_3, p_4, p_5 respectively. What is the corresponding reliability function?

SOLUTION:

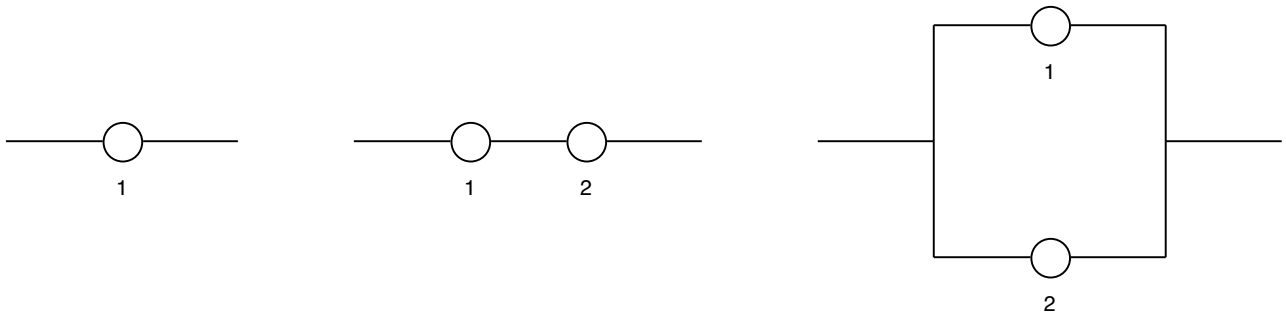
$$\begin{aligned} h(\boldsymbol{p}) &= E[\phi(\boldsymbol{X})] = E[X_1 \cdot X_2 \cdot (X_3 \amalg X_4) \cdot X_5] \\ &= E[X_1] \cdot E[X_2] \cdot (E[X_3] \amalg E[X_4]) \cdot E[X_5] \quad \text{using independence} \\ &= p_1 \cdot p_2 \cdot (p_3 \amalg p_4) \cdot p_5 \end{aligned}$$



Exercise 2.5

There are 8 different coherent systems of order less than or equal to 3 (not counting permutations in the numbering of components). What are they?

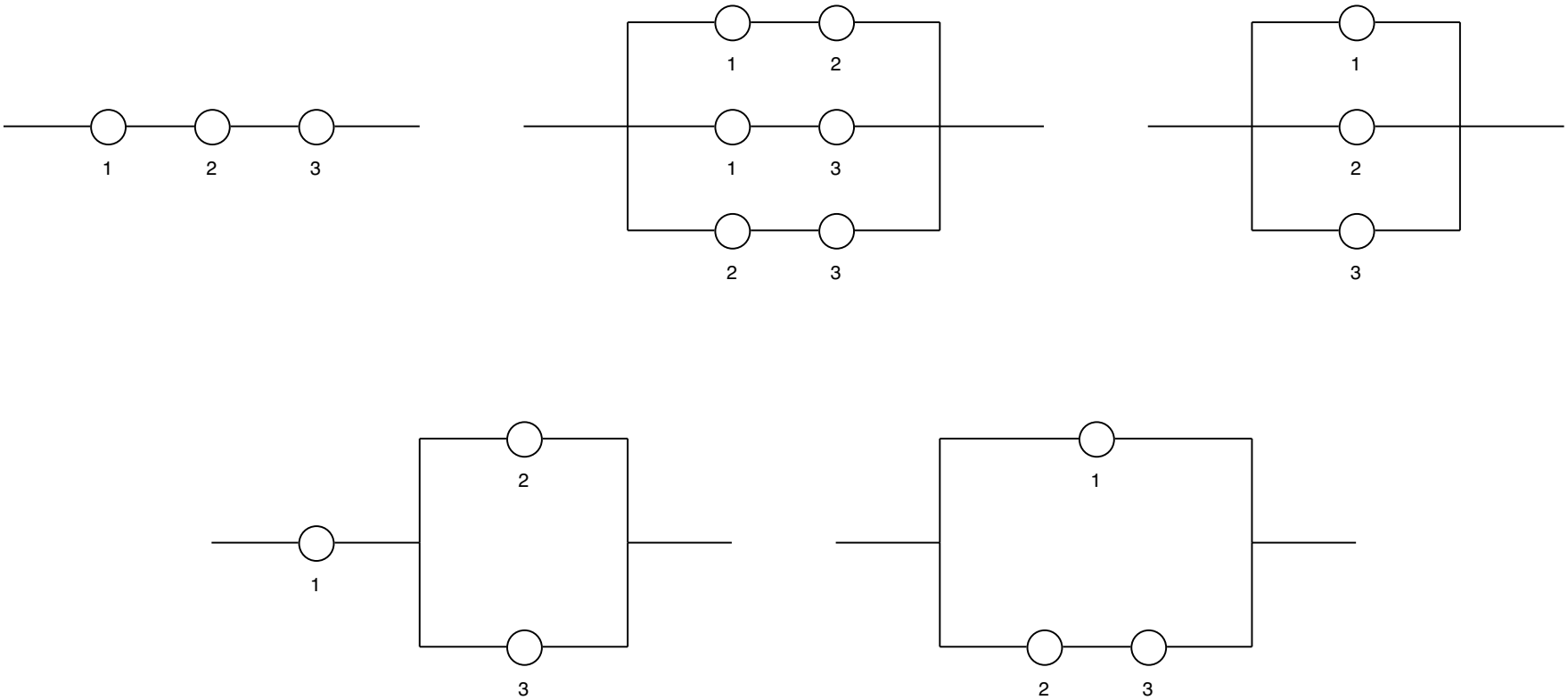
SOLUTION: There are three coherent systems of order 1 or 2:





Exercise 2.5 (cont.)

There are five coherent systems of order 3:



Exercise 2.6

Consider a monotone system (C, ϕ) of order n, and let $A \subset C$ be the set of irrelevant components.

Furthermore, let $(C \setminus A, \phi')$, be a binary monotone system of order $m = n - |A|$, where ϕ' is defined for all m-dimensional binary vectors $\boldsymbol{x} \in \{0,1\}^m$ as:

$$\phi'(\boldsymbol{x}) = \phi(\mathbf{1}^A, \boldsymbol{x}^{C \setminus A})$$

Show that $(C \setminus A, \phi')$ is coherent.

SOLUTION: Since ϕ is non-decreasing in each argument, it follows that $\phi(\mathbf{1}^A, \boldsymbol{x}^{C \setminus A})$ is non-decreasing in x_i for all $i \in C \setminus A$.

Hence, $(C \setminus A, \phi')$ is indeed a binary monotone system.



Exercise 2.6 (cont.)

We then let $i \in (C \setminus A)$. By assumption i is relevant in (C, ϕ). That is, there exists a $(\cdot_i, \boldsymbol{x})$ such that:

$$\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x}) = 1.$$

This equation can also be written as:

$$\phi(1_i, \boldsymbol{x}^A, \boldsymbol{x}^{C \setminus A}) - \phi(0_i, \boldsymbol{x}^A, \boldsymbol{x}^{C \setminus A}) = 1.$$

Since all the components in A are irrelevant in the original system, we may replace $\boldsymbol{x}^A$ by $\mathbf{1}^A$ without changing the value of ϕ:

$$\phi(1_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A}) - \phi(0_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A}) = 1.$$

Hence, we have shown that there exists a vector $(\cdot_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A})$ such that:

$$\phi(1_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A}) - \phi(0_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A}) = 1.$$

Thus, we conclude that i is relevant in $(C \setminus A, \phi')$, and since this holds for all $i \in C \setminus A$, we conclude that $(C \setminus A, \phi')$ is coherent.



Exercise 2.7

Let (C, ϕ) be a non-trivial binary monotone system of order n. Then for all $\boldsymbol{x} \in \{0,1\}^n$ we have:

$$\prod_{i=1}^{n} x_i \leq \phi(\boldsymbol{x}) \leq \coprod_{i=1}^{n} x_i.$$

Prove the right-hand inequality.

SOLUTION: If $\coprod_{i=1}^{n} x_i = 1$, this inequality is trivial since $\phi(\boldsymbol{x}) \in \{0,1\}$ for all $\boldsymbol{x} \in \{0,1\}^n$.

If on the other hand $\coprod_{i=1}^{n} x_i = 0$, we must have $\boldsymbol{x} = \mathbf{0}$.

Since (C, ϕ) is assumed to be non-trivial, it follows that $\phi(\mathbf{0}) = 0$. Thus, the inequality is valid in this case as well.

This completes the proof of the right-hand inequality.



Exercise 2.8

Let (C, ϕ) be a binary monotone system of order n. Show that for all $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^n$ we have:

$$\phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y})$$

Moreover, assume that (C, ϕ) is coherent. Prove that equality holds for all $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^n$ if and only if (C, ϕ) is a series system.

SOLUTION: Since $\boldsymbol{x} \cdot \boldsymbol{y} \leq \boldsymbol{x}$ and $\boldsymbol{x} \cdot \boldsymbol{y} \leq \boldsymbol{y}$ and ϕ is non-decreasing in each argument, we have:

$$\phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \phi(\boldsymbol{x}) \text{ and } \phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \phi(\boldsymbol{y}).$$

Hence, we have:

$$\phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \min\{\phi(\boldsymbol{x}), \phi(\boldsymbol{y})\} = \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y}).$$



Exercise 2.8 (cont.)

It remains to prove that if (C, ϕ) is coherent, then $\phi(\boldsymbol{x} \cdot \boldsymbol{y}) = \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y})$ if and only if (C, ϕ) is a series system.

SOLUTION: Assume first that (C, ϕ) is a series system. Then:

$$\phi(\boldsymbol{x} \cdot \boldsymbol{y}) = \prod_{i=1}^{n}(x_i \cdot y_i) = [\prod_{i=1}^{n} x_i] \cdot [\prod_{i=1}^{n} y_i] = \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y}).$$

Assume conversely that $\phi(\boldsymbol{x} \cdot \boldsymbol{y}) = \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y})$, and choose $i \in C$ arbitrarily.

Since (C, ϕ) is coherent, there exists a vector $(\cdot_i, \boldsymbol{x})$ such that:

$$\phi(1_i, \boldsymbol{x}) = 1 \text{ and } \phi(0_i, \boldsymbol{x}) = 0.$$



Exercise 2.8 (cont.)

For this particular $(\cdot_i, \boldsymbol{x})$ we have by the assumption that:

$$\begin{aligned} 0 = \phi(0_i, \boldsymbol{x}) &= \phi((0_i, \mathbf{1}) \cdot (1_i, \boldsymbol{x})) \\ &= \phi(0_i, \mathbf{1}) \cdot \phi(1_i, \boldsymbol{x}) = \phi(0_i, \mathbf{1}) \cdot 1 \end{aligned}$$

Hence, $\phi(0_i, \mathbf{1}) = 0$, and since obviously $\phi(1_i, \mathbf{1}) = 1$, we conclude that:

$$\phi(x_i, \mathbf{1}) = x_i, \text{ for } x_i = 0, 1.$$

Since $i \in C$ was chosen arbitrarily, we must have:

$$\phi(x_i, \mathbf{1}) = x_i, \text{ for } x_i = 0, 1, \text{ for all } i \in C.$$



Exercise 2.8 (cont.)

By repeated use of the assumption, we get:

$$\begin{aligned}\phi(\boldsymbol{x}) &= \phi((x_1, \mathbf{1}) \cdot (x_2, \mathbf{1}) \cdots (x_n, \mathbf{1})) \\ &= \phi(x_1, \mathbf{1}) \cdot \phi(x_2, \mathbf{1}) \cdots \phi(x_n, \mathbf{1}) \\ &= x_1 \cdot x_2 \cdots x_n = \prod_{i=1}^{n} x_i\end{aligned}$$

Thus, we conclude that (C, ϕ) is a series system.



Exercise 2.9

Prove that the dual system of a k-out-of-n system is an $(n-k+1)$-out-of-n system.

SOLUTION: Assume that (C, ϕ) is a k-out-of-n-system. The structure function, ϕ, can then be written as:

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{otherwise.} \end{cases}$$

More compactly we may write this as:

$$\phi(\boldsymbol{x}) = I(\sum_{i=1}^{n} x_i \geq k).$$



Exercise 2.9 (cont.)

We then have:

$$\phi^D(\boldsymbol{x}) = 1 - \phi(\mathbf{1} - \boldsymbol{x}) = 1 - I(\sum_{i=1}^{n}(1 - x_i) \geq k)$$

$$= 1 - I(n - \sum_{i=1}^{n} x_i \geq k) = 1 - I(\sum_{i=1}^{n} x_i \leq n - k)$$

$$= I(\sum_{i=1}^{n} x_i > n - k) = I(\sum_{i=1}^{n} x_i \geq n - k + 1)$$

Hence, (C, ϕ^D) is an $(n - k + 1)$-out-of-n system.



Chapter 3

Basic reliability calculation methods



Pivotal decompositions



Pivotal decompositions

Theorem

Let (C, ϕ) be a binary monotone system. We then have:

$$\phi(\boldsymbol{x}) = x_i \phi(1_i, \boldsymbol{x}) + (1 - x_i)\phi(0_i, \boldsymbol{x}), \quad i \in C. \tag{1}$$

Similarly, for the reliability function of a binary monotone system where the component state variables are independent, we have

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i)h(0_i, \boldsymbol{p}), \quad i \in C. \tag{2}$$



Pivotal decompositions (cont.)

PROOF: Let $i \in C$, and consider two cases:

CASE 1. $x_i = 1$. Then the right-hand side of (1) becomes:

$$x_i \phi(1_i, \boldsymbol{x}) + (1 - x_i)\phi(0_i, \boldsymbol{x}) = \phi(1_i, \boldsymbol{x}).$$

Hence, $\phi(\boldsymbol{x}) = \phi(1_i, \boldsymbol{x})$, so (1) holds in this case.

CASE 2. $x_i = 0$. Then the right-hand side of (1) becomes:

$$x_i \phi(1_i, \boldsymbol{x}) + (1 - x_i)\phi(0_i, \boldsymbol{x}) = \phi(0_i, \boldsymbol{x}).$$

Hence, $\phi(\boldsymbol{x}) = \phi(0_i, \boldsymbol{x})$, so (1) holds in this case as well.

Equation (2) is proved by replacing the vector $\boldsymbol{x}$ by $\boldsymbol{X}$ in (1), and taking the expectation.



Pivotal decompositions (cont.)

EXAMPLE: Let (C, ϕ) be a 2-out-of-3 system where the component states are independent with reliabilities p_1, p_2, p_3, We then have:

$$\begin{aligned}\phi(\boldsymbol{x}) &= x_1\phi(1_1, \boldsymbol{x}) + (1 - x_1)\phi(0_1, \boldsymbol{x}) \\ &= x_1[x_2 \amalg x_3] + (1 - x_1)[x_2 \cdot x_3] \\ &= x_1[x_2 + x_3 - x_2 \cdot x_3] + (1 - x_1)[x_2 \cdot x_3]\end{aligned}$$

From this it follows that:

$$\begin{aligned}h(\boldsymbol{p}) &= p_1[p_2 + p_3 - p_2 \cdot p_3] + (1 - p_1)[p_2 \cdot p_3] \\ &= p_1p_2 + p_1p_3 - p_1p_2p_3 + p_2p_3 - p_1p_2p_3 \\ &= p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3\end{aligned}$$



Pivotal decompositions (cont.)

Corollary

Let (C, ϕ) be a binary monotone system. We then have:

$$\phi(\boldsymbol{x}) = x_i x_j \phi(1_i, 1_j, \boldsymbol{x}) + x_i(1 - x_j)\phi(1_i, 0_j, \boldsymbol{x}) + (1 - x_i)x_j\phi(0_i, 1_j, \boldsymbol{x}) + (1 - x_i)(1 - x_j)\phi(0_i, 0_j, \boldsymbol{x}), \quad i, j \in C.$$

Similarly, for the reliability function of a binary monotone system where the component state variables are independent, we have:

$$h(\boldsymbol{p}) = p_i p_j h(1_i, 1_j, \boldsymbol{p}) + p_i(1 - p_j)h(1_i, 0_j, \boldsymbol{p}) + (1 - p_i)p_j h(0_i, 1_j, \boldsymbol{p}) + (1 - p_i)(1 - p_j)h(0_i, 0_j, \boldsymbol{p}), \quad i, j \in C.$$

PROOF: Use the pivotal decomposition theorem. Then apply the same theorem to $\phi(1_i, \boldsymbol{x})$ and $\phi(0_i, \boldsymbol{x})$.



Series and parallel components

Definition

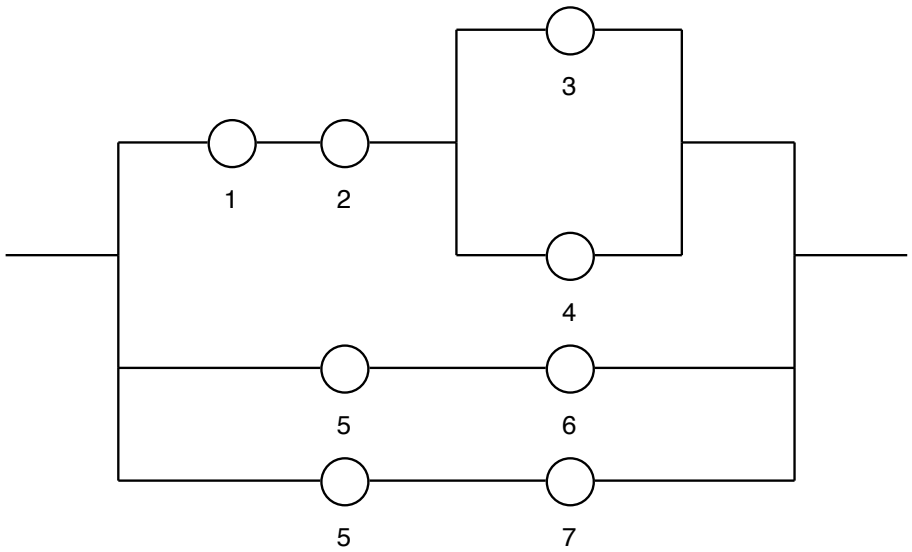
Let (C, ϕ) be a binary monotone system, and let $i, j \in C$.

We say that i and j are *in series* if ϕ depends on the component state variables, x_i and x_j, only through the product $x_i \cdot x_j$.

We say that i and j are *in parallel* if ϕ depends on the component state variables, x_i and x_j, only through the coproduct $x_i \amalg x_j$.



Series and parallel components (cont.)



In this system components 1 and 2 are in series, while components 3 and 4 are in parallel. Note, however, that components 5 and 6 are *not* in series since component 5 is also connected via component 7. Moreover, components 6 and 7 are in parallel.



Series and parallel components (cont.)

NOTE: Let $\boldsymbol{x}^{C\setminus\{i,j\}}$ denote the subvector of $\boldsymbol{x}$ containing all component state variables except x_i and x_j.

If i and j are *in series*, the structure function ϕ can be written as:

$$\phi(\boldsymbol{x}) = \psi(x_i \cdot x_j, \boldsymbol{x}^{C\setminus\{i,j\}})$$

for some binary non-decreasing function ψ.

This implies that:

$$\phi(0_i, 0_j, \boldsymbol{x}) = \phi(1_i, 0_j, \boldsymbol{x}) = \phi(0_i, 1_j, \boldsymbol{x}) = \psi(0, \boldsymbol{x}^{C\setminus\{i,j\}})$$



Series and parallel components (cont.)

Similarly, if *i* and *j* are *in parallel*, the structure function ϕ can be written as:

$$\phi(\boldsymbol{x}) = \psi(x_i \amalg x_j, \boldsymbol{x}^{C\setminus\{i,j\}})$$

for some binary non-decreasing function ψ.

This implies that:

$$\phi(1_i, 0_j, \boldsymbol{x}) = \phi(0_i, 1_j, \boldsymbol{x}) = \phi(1_i, 1_j, \boldsymbol{x}) = \psi(1, \boldsymbol{x}^{C\setminus\{i,j\}})$$



Series and parallel components (cont.)

Theorem

Let (C, ϕ) be a binary monotone system, and let $i, j \in C$. Moreover, assume that the component state variables are independent.

If i and j are in series, then the reliability function, h, depends on p_i and p_j only through $p_i \cdot p_j$.

If i and j are in parallel, then the reliability function, h, depends on p_i and p_j only through $p_i \amalg p_j$.

NOTE: We have:

$$x_i x_j + x_i(1 - x_j) + (1 - x_i)x_j + (1 - x_i)(1 - x_j)$$

$$= x_i x_j + x_i - x_i x_j + x_j - x_i x_j + 1 - x_i - x_j + x_i x_j = 1$$



Series and parallel components (cont.)

PROOF: If i and j are in series, we know that:

$$\phi(1_i, 0_j, \boldsymbol{x}) = \phi(0_i, 1_j, \boldsymbol{x}) = \phi(0_i, 0_j, \boldsymbol{x}).$$

Thus, by pivotal decomposition we have:

$$\begin{aligned}\phi(\boldsymbol{x}) &= x_i x_j \phi(1_i, 1_j, \boldsymbol{x}) + x_i(1 - x_j)\phi(1_i, 0_j, \boldsymbol{x}) \\ &\quad + (1 - x_i)x_j \phi(0_i, 1_j, \boldsymbol{x}) + (1 - x_i)(1 - x_j)\phi(0_i, 0_j, \boldsymbol{x}) \\ &= (x_i x_j) \cdot \phi(1_i, 1_j, \boldsymbol{x}) + (1 - (x_i x_j)) \cdot \phi(0_i, 0_j, \boldsymbol{x}).\end{aligned}$$

Hence, by replacing the vector $\boldsymbol{x}$ by $\boldsymbol{X}$ and taking expectations we get:

$$h(\boldsymbol{p}) = (p_i p_j) \cdot h(1_i, 1_j, \boldsymbol{p}) + (1 - (p_i p_j)) \cdot h(0_i, 0_j, \boldsymbol{p}).$$

That is, h, depends on p_i and p_j only through $p_i \cdot p_j$.



Series and parallel components (cont.)

If i and j are in parallel, we know that:

$$\phi(1_i, 1_j, \boldsymbol{x}) = \phi(1_i, 0_j, \boldsymbol{x}) = \phi(0_i, 1_j, \boldsymbol{x}).$$

Thus, by pivotal decomposition we have:

$$\begin{aligned}\phi(\boldsymbol{x}) &= x_i x_j \phi(1_i, 1_j, \boldsymbol{x}) + x_i(1 - x_j)\phi(1_i, 0_j, \boldsymbol{x}) \\ &\quad + (1 - x_i)x_j \phi(0_i, 1_j, \boldsymbol{x}) + (1 - x_i)(1 - x_j)\phi(0_i, 0_j, \boldsymbol{x}) \\ &= (x_i \amalg x_j) \cdot \phi(1_i, 1_j, \boldsymbol{x}) + (1 - (x_i \amalg x_j)) \cdot \phi(0_i, 0_j, \boldsymbol{x}).\end{aligned}$$

Hence, by replacing the vector $\boldsymbol{x}$ by $\boldsymbol{X}$ and taking expectations we get:

$$h(\boldsymbol{p}) = (p_i \amalg p_j) \cdot h(1_i, 1_j, \boldsymbol{p}) + (1 - (p_i \amalg p_j)) \cdot h(0_i, 0_j, \boldsymbol{p}).$$

That is, h, depends on p_i and p_j only through $p_i \amalg p_j$.



s-p-reductions

Consider a binary monotone system, (C, ϕ) where the component state variables are independent, and let $i, j \in C$.

SERIES REDUCTION: If the components i and j are in series, then we may replace i and j by a single component i' with reliability $p_{i'} = p_i p_j$ without altering the system reliability.

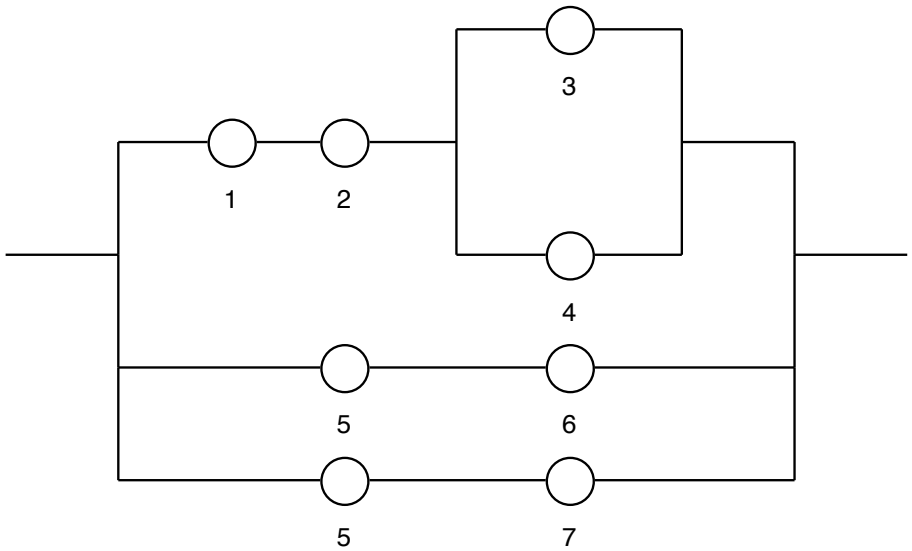
PARALLEL REDUCTION: If the components i and j are in parallel, then we may replace i and j by a single component i' with reliability $p_{i'} = p_i \amalg p_j$ without altering the system reliability.

Series and parallel reductions are referred to as *s-p-reductions*. Each s-p-reduction reduces the number of components in the system by one.

A system that can be reduced to a single component by applying a sequence of s-p-reductions is called an *s-p-system*.



s-p-reductions (cont.)



This 7-component system is an s-p-system. Its reliability function can be derived using s-p-reductions *only* and is given by:

$$h(\boldsymbol{p}) = [p_1 p_2 (p_3 \amalg p_4)] \amalg [p_5 (p_6 \amalg p_7)]$$

