STK3405 – Lecture 5

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Section 3.3

Modules of monotone systems



Modules of monotone systems

If $A \subseteq C$, then the complement set of A, i.e., $C \setminus A$, is denoted by $\bar{A}$. We also let $\boldsymbol{x}^A$ and $\boldsymbol{x}^{\bar{A}}$ denote the subvectors of $\boldsymbol{x}$ corresponding to the sets A and $\bar{A}$ respectively.

Definition

Let (C, ϕ) be a binary monotone system, and $A \subseteq C$. The monotone system (A, χ) is a *module* of (C, ϕ) if and only if the structure function ϕ can be written as:

$$\phi(\boldsymbol{x}) = \psi(\chi(\boldsymbol{x}^A), \boldsymbol{x}^{\bar{A}}), \qquad \text{for all } \boldsymbol{x} \in \{0,1\}^n,$$

where ψ is a monotone structure function. The set A is called a *modular set* of (C, ϕ).



Modules of monotone systems

Definition

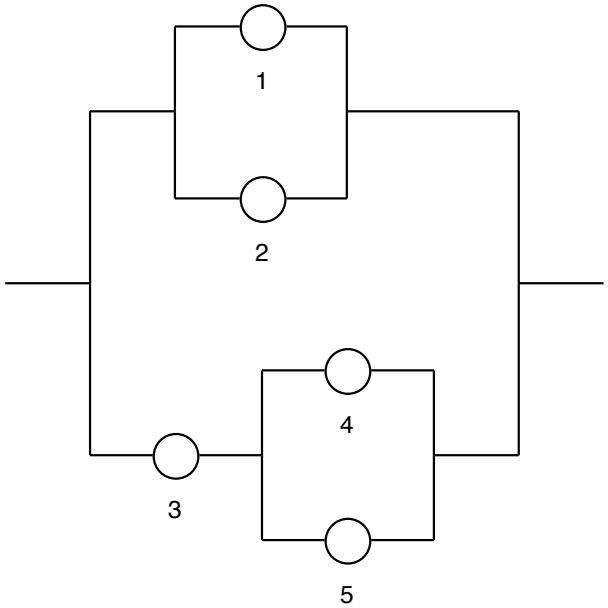
A *modular decomposition* of a monotone system (C, ϕ) is a set of modules $\{(A_j, \chi_j)\}_{j=1}^{r}$ connected by a binary monotone organisation structure function ψ. The following conditions must be satisfied:

- $C = \bigcup_{j=1}^{r} A_j$, and $A_j \cap A_k = \emptyset$ for $j \neq k$.
- $\phi(\boldsymbol{x}) = \psi[\chi_1(\boldsymbol{x}^{A_1}), \ldots, \chi_r(\boldsymbol{x}^{A_r})]$.

We observe that a modular decomposition is a disjoint partition of the component set into modules such that the structure function of the whole system is a function of the structure functions of these modules.



Modules of monotone systems (cont.)



Modules: (A_1, χ_1) and (A_2, χ_2) where $A_1 = \{1, 2\}$ and $A_2 = \{3, 4, 5\}$, and:

$$\chi_1(x_1, x_2) = x_1 \amalg x_2,$$

$$\chi_2(x_3, x_4, x_5) = x_3 \cdot (x_4 \amalg x_5)$$

$$\psi(\chi_1, \chi_2) = \chi_1 \amalg \chi_2$$



Exercise 3.2

Prove Theorem 3.2.5:

Theorem

Let (C, ϕ) be a binary monotone system, and let (C, ϕ^D) be its dual. Then the following statements hold:

- $\boldsymbol{x}$ *is a path vector (alternatively, cut vector) for (C, ϕ) if and only if $\boldsymbol{x}^D$ is a cut vector (path vector) for (C, ϕ^D).*
- *A minimal path set (alternatively, cut set) for (C, ϕ) is a minimal cut set (path set) for (C, ϕ^D).*



Exercise 3.2 (cont.)

PROOF: Assume that $\boldsymbol{x}$ is a *path vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 1$ and we get:

$$\phi^D(\boldsymbol{x}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{x}^D) = 1 - \phi(\boldsymbol{x}) = 1 - 1 = 0.$$

Hence, $\boldsymbol{x}^D$ is a *cut vector* for (C, ϕ^D).

Similarly, assume that $\boldsymbol{x}$ is a *cut vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 0$ and we get:

$$\phi^D(\boldsymbol{x}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{x}^D) = 1 - \phi(\boldsymbol{x}) = 1 - 0 = 1.$$

Hence, $\boldsymbol{x}^D$ is a *path vector* for (C, ϕ^D).



Exercise 3.2 (cont.)

Assume that $\boldsymbol{x}$ is a *minimal path vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 1$ and $\phi(\boldsymbol{y}) = 0$ for all $\boldsymbol{y} < \boldsymbol{x}$. We have already shown that $\phi^D(\boldsymbol{x}^D) = 0$.

We then choose $\boldsymbol{y}^D > \boldsymbol{x}^D$, and note that this implies that $\boldsymbol{y} < \boldsymbol{x}$. Thus, we get: we have:

$$\phi^D(\boldsymbol{y}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{y}^D) = 1 - \phi(\boldsymbol{y}) = 1 - 0 = 1.$$

Hence, $\boldsymbol{x}^D$ is a *minimal cut vector* for (C, ϕ^D).

Similarly, assume that $\boldsymbol{x}$ is a *minimal cut vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 0$ and $\phi(\boldsymbol{y}) = 1$ for all $\boldsymbol{y} > \boldsymbol{x}$. We have already shown that $\phi^D(\boldsymbol{x}^D) = 1$.

We then choose $\boldsymbol{y}^D < \boldsymbol{x}^D$, and note that this implies that $\boldsymbol{y} > \boldsymbol{x}$. Thus, we get:

$$\phi^D(\boldsymbol{y}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{y}^D) = 1 - \phi(\boldsymbol{y}) = 1 - 1 = 0.$$

Hence, $\boldsymbol{x}^D$ is a *minimal path vector* for (C, ϕ^D).



Exercise 3.2 (cont.)

Assume that $P \subseteq C$ is a *minimal path set* for (C, ϕ), and let $\boldsymbol{x}$ be the corresponding *minimal path vector*.

We have already shown that this implies that $\boldsymbol{x}^D = \mathbf{1} - \boldsymbol{x}$ is a minimal cut vector for (C, ϕ^D). Hence, it follows that:

$$K^D = \{i : x_i^D = 0\} = \{i : x_i = 1\} = P$$

is a *minimal cut set* for (C, ϕ^D).

Assume that $K \subseteq C$ is a *minimal cut set* for (C, ϕ), and let $\boldsymbol{x}$ be the corresponding *minimal cut vector*.

We have already shown that this implies that $\boldsymbol{x}^D = \mathbf{1} - \boldsymbol{x}$ is a minimal path vector for (C, ϕ^D). Hence, it follows that:

$$P^D = \{i : x_i^D = 1\} = \{i : x_i = 0\} = K$$

is a *minimal path set* for (C, ϕ^D).



Exercise 3.4

Let (C, ϕ) be a binary monotone system with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$.

We then recall the formula:

$$\phi(\boldsymbol{x}) = \coprod_{j=1}^{p} \prod_{i \in P_j} x_i = \prod_{j=1}^{k} \coprod_{i \in K_j} x_i$$

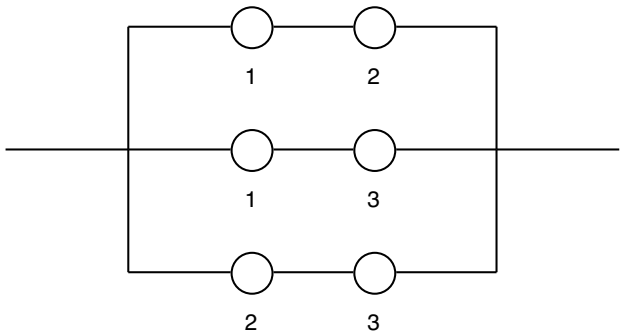
In this exercise we will apply this formula to various binary monotone systems.



Exercise 3.4 (cont.)

Find the representations via the minimal path sets and the minimal cut sets:

(i) **2-out-3 system:**



Minimal path sets: $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$, $P_3 = \{2, 3\}$.

$$\phi(\boldsymbol{X}) = (X_1 \cdot X_2) \amalg (X_1 \cdot X_3) \amalg (X_2 \cdot X_3)$$

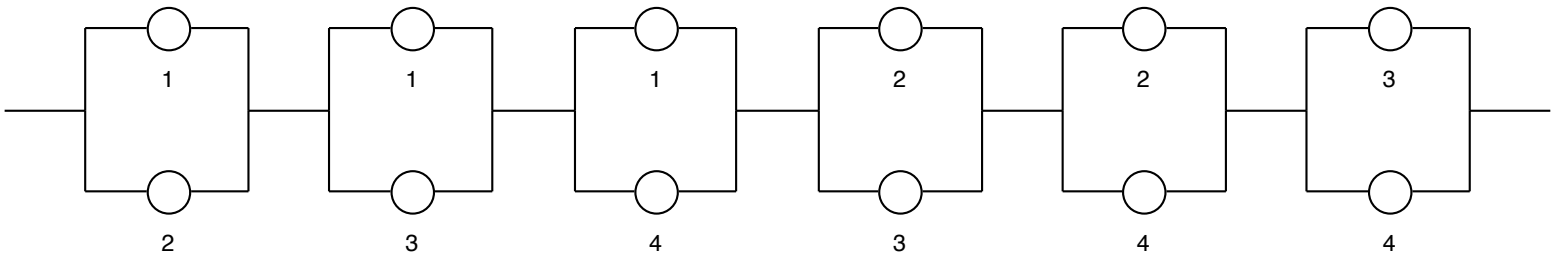
Minimal cut sets: $K_1 = \{1, 2\}$, $K_2 = \{1, 3\}$, $K_3 = \{2, 3\}$.

$$\phi(\boldsymbol{X}) = (X_1 \amalg X_2) \cdot (X_1 \amalg X_3) \cdot (X_2 \amalg X_3)$$



Exercise 3.4 (cont.)

(ii) **3-out-4 system:**



Minimal path sets:
$P_1 = \{1,2,3\}$, $P_2 = \{1,2,4\}$, $P_3 = \{1,3,4\}$, $P_4 = \{2,3,4\}$.

$$\phi(\boldsymbol{X}) = (X_1 \cdot X_2 \cdot X_3) \amalg (X_1 \cdot X_2 \cdot X_4) \amalg (X_1 \cdot X_3 \cdot X_4) \amalg (X_2 \cdot X_3 \cdot X_4)$$

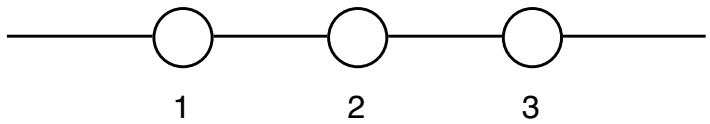
Minimal cut sets:
$K_1 = \{1,2\}$, $K_2 = \{1,3\}$, $K_3 = \{1,4\}$, $K_4 = \{2,3\}$, $K_5 = \{2,4\}$, $K_6 = \{3,4\}$.

$$\phi(\boldsymbol{X}) = (X_1 \amalg X_2) \cdot (X_1 \amalg X_3) \cdot (X_1 \amalg X_4) \cdot (X_2 \amalg X_3) \cdot (X_2 \amalg X_4) \cdot (X_3 \amalg X_4)$$



Exercise 3.4 (cont.)

(iii) **Series system of** 3 **components:**



Minimal path set: $P = \{1, 2, 3\}$.

$$\phi(\boldsymbol{X}) = X_1 \cdot X_2 \cdot X_3$$

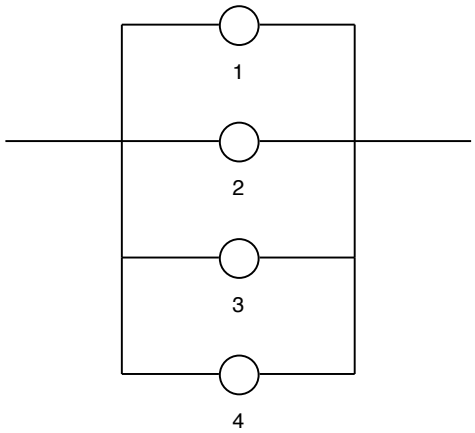
Minimal cut sets: $K_1 = \{1\}$, $K_2 = \{2\}$, $K_3 = \{3\}$.

$$\phi(\boldsymbol{X}) = X_1 \cdot X_2 \cdot X_3$$



Exercise 3.4 (cont.)

(iv) **Parallel system of** 4 **components:**



Minimal path set: $P_1 = \{1\}$, $P_2 = \{2\}$, $P_3 = \{3\}$, $P_4 = \{4\}$.

$$\phi(\boldsymbol{X}) = X_1 \amalg X_2 \amalg X_3 \amalg X_4$$

Minimal cut sets: $K = \{1, 2, 3, 4\}$.

$$\phi(\boldsymbol{X}) = X_1 \amalg X_2 \amalg X_3 \amalg X_4$$



Exercise 3.5

Consider a coherent system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$. Prove that

$$\bigcup_{j=1}^{p} P_j = \bigcup_{j=1}^{k} K_j = C.$$

NOTE: Since all the minimal path and cut sets are subsets of the component set C, we obviously have that:

$$\bigcup_{j=1}^{p} P_j \subseteq C \text{ and } \bigcup_{j=1}^{k} K_j \subseteq C$$

In order to show that for coherent systems the three sets are equal we show the following related result.



Exercise 3.5 (cont.)

Theorem

Let (C, ϕ) be a binary monotone system. Then the following three statements are equivalent:

- *$i \in C$ is relevant*
- *$i \in P$ for at least one minimal path set P*
- *$i \in K$ for at least one minimal cut set K*

PROOF: Assume that $i \in C$ is relevant. Then there exists a vector $(\cdot_i, \boldsymbol{x})$ such that:

$$\phi(1_i, \boldsymbol{x}) = 1 \quad \text{and} \quad \phi(0_i, \boldsymbol{x}) = 0 \qquad (*)$$

NOTE: We can always choose $(\cdot_i, \boldsymbol{x})$ such that it is a *minimal* vector with the property $(*)$ in the sense that if $(\cdot_i, \boldsymbol{y}) < (\cdot_i, \boldsymbol{x})$, then $\phi(1_i, \boldsymbol{y}) = \phi(0_i, \boldsymbol{y}) = 0$.



Exercise 3.5 (cont.)

It then follows that $(1_i, \boldsymbol{x})$ is a minimal path vector since $\boldsymbol{y} < (1_i, \boldsymbol{x})$ implies that $\phi(\boldsymbol{y}) = 0$.

We then let $P = \{j \in C : x_j = 1\}$, which by definition this is a minimal path set. Moreover, $i \in P$.

Thus, we have shown that if $i \in C$ is relevant, then $i \in P$ for at least one minimal path set P.

We now show the converse implication, i.e., that if $i \in P$ for at least one minimal path set P, then $i \in C$ is relevant.



Exercise 3.5 (cont.)

Assume that there exists a minimal path set P such that $i \in P$, and let $\boldsymbol{x}$ be the corresponding minimal path vector. Then by definition:

$$x_i = 1 \quad \text{and} \quad \phi(\boldsymbol{x}) = \phi(1_i, \boldsymbol{x}) = 1.$$

Moreover, if $\boldsymbol{y} < \boldsymbol{x}$, then $\phi(\boldsymbol{y}) = 0$. Since in particular $(0_i, \boldsymbol{x}) < \boldsymbol{x}$, it follows that $\phi(0_i, \boldsymbol{x}) = 0$, i.e., i is relevant.

Thus, we have shown that if $i \in P$ for at least one minimal path set P, then $i \in C$ is relevant.

The proof that a component $i \in C$ is relevant if and only if $i \in K$ for at least one minimal cut set K is proved in a similar way.



Exercise 3.5 (cont.)

A direct consequence of the theorem we just proved, is:

Corollary

Consider a binary monotone system (C, ϕ) *with minimal path sets* $P_1, \ldots, P_p$ *and minimal cut sets* $K_1, \ldots, K_k$*. Then the following are equivalent:*

1. (C, ϕ) *is coherent*
2. $\bigcup_{j=1}^{p} P_j = C$
3. $\bigcup_{j=1}^{k} K_j = C$

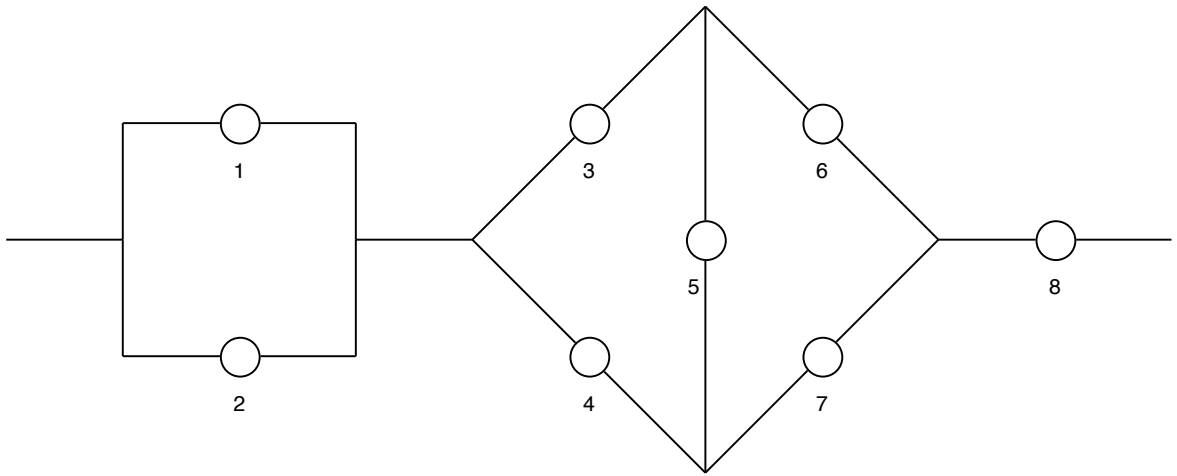
PROOF: Assume that (1) holds true. Hence, by the theorem every component is contained in at least one minimal path set and in at least one minimal cut set. Hence, (1) implies both (2) and (3).

Assume that (2) [or (3)] holds true. Hence, by the theorem all components are relevant. Hence, (2) [and (3)] implies (1).



Exercise 3.6

Find the minimal path sets and the minimal cut sets of the system in the system below.



Minimal path sets:

$P_1 = \{1, 3, 6, 8\}$, $P_2 = \{1, 3, 5, 7, 8\}$, $P_3 = \{1, 4, 5, 6, 8\}$, $P_4 = \{1, 4, 7, 8\}$,
$P_5 = \{2, 3, 6, 8\}$, $P_6 = \{2, 3, 5, 7, 8\}$, $P_7 = \{2, 4, 5, 6, 8\}$, $P_8 = \{2, 4, 7, 8\}$.

Minimal cut sets:

$K_1 = \{1, 2\}$, $K_2 = \{3, 4\}$, $K_3 = \{3, 5, 7\}$, $K_4 = \{4, 5, 6\}$, $K_5 = \{6, 7\}$, $K_6 = \{8\}$



Exercise 3.6 (cont.)

Find two different expressions for the structure function of this system.

$$\phi(\boldsymbol{X}) = \coprod_{j=1}^{8} \prod_{i \in P_j} X_i$$

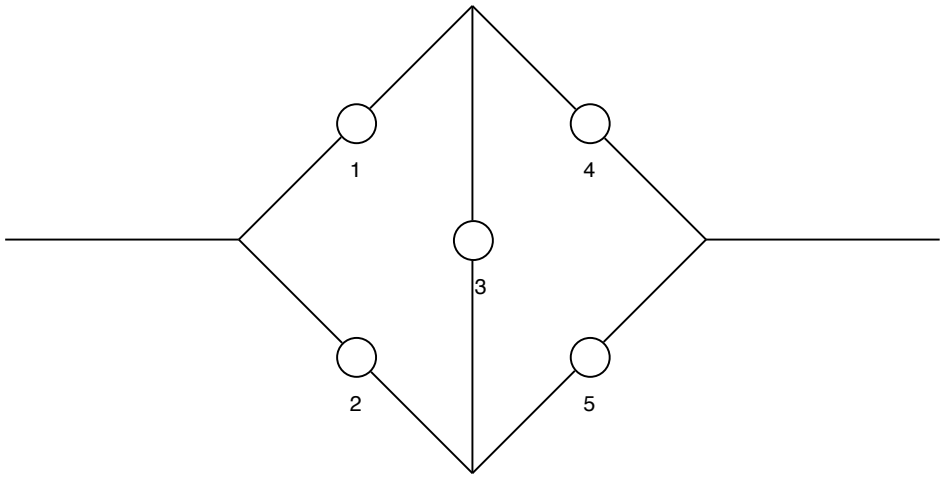
$$\phi(\boldsymbol{X}) = \prod_{j=1}^{6} \coprod_{i \in K_j} X_i$$

$$\begin{aligned}\phi(\boldsymbol{X}) = (X_1 \amalg X_2) \cdot [X_5 \cdot (X_3 \amalg X_4) \cdot (X_6 \amalg X_7) \\ + (1 - X_5) \cdot ((X_3 X_6) \amalg (X_4 X_7))] \cdot X_8\end{aligned}$$



Exercise 3.7

Show that if (C, ϕ) is a bridge structure, then (C, ϕ^D) is a bridge structure as well.



Minimal path sets:

$P_1 = \{1, 4\}$, $P_2 = \{1, 3, 5\}$, $P_3 = \{2, 3, 4\}$, $P_4 = \{2, 5\}$

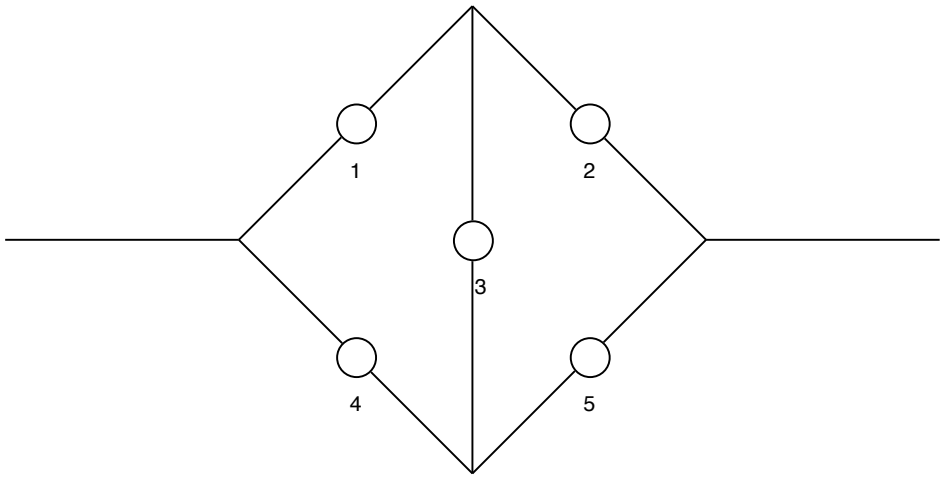
Minimal cut sets:

$K_1 = \{1, 2\}$, $K_2 = \{1, 3, 5\}$, $K_3 = \{2, 3, 4\}$, $K_4 = \{4, 5\}$.



Exercise 3.7 (cont.)

The dual system (C, ϕ^D):



Minimal path sets:

$P_1 = \{1, 2\}$, $P_2 = \{1, 3, 5\}$, $P_3 = \{2, 3, 4\}$, $P_4 = \{4, 5\}$

Minimal cut sets:

$K_1 = \{1, 4\}$, $K_2 = \{1, 3, 5\}$, $K_3 = \{2, 3, 4\}$, $K_4 = \{2, 5\}$.



Exercise 3.8

Let (A, χ) be a module of (C, ϕ). Assume that $\boldsymbol{x}_1$ and $\boldsymbol{x}_0$ are such that $\chi(\boldsymbol{x}_1^A) = 1$ and $\chi(\boldsymbol{x}_0^A) = 0$. Prove that for all $(\cdot^A, \boldsymbol{x}^{\bar{A}})$ we have:

$$\phi(\boldsymbol{x}_1^A, \boldsymbol{x}_1^{\bar{A}}) = \phi(\boldsymbol{1}^A, \boldsymbol{x}_1^{\bar{A}}) \quad \text{and} \quad \phi(\boldsymbol{x}_0^A, \boldsymbol{x}_0^{\bar{A}}) = \phi(\boldsymbol{0}^A, \boldsymbol{x}_0^{\bar{A}}).$$

PROOF: Let ψ be the organising structure function for ϕ and χ. That is, we have:

$$\phi(\boldsymbol{x}) = \psi(\chi(\boldsymbol{x}^A), \boldsymbol{x}^{\bar{A}}) \quad \text{for all } \boldsymbol{x}.$$

Hence, since χ is non-decreasing we have by the assumptions that:

$$\begin{aligned}\phi(\boldsymbol{x}_1) &= \psi(\chi(\boldsymbol{x}_1^A), \boldsymbol{x}_1^{\bar{A}}) = \psi(1, \boldsymbol{x}_1^{\bar{A}}) \\ &= \psi(\chi(\boldsymbol{1}^A), \boldsymbol{x}_1^{\bar{A}}) = \phi(\boldsymbol{1}^A, \boldsymbol{x}_1^{\bar{A}}).\end{aligned}$$

Similarly:

$$\begin{aligned}\phi(\boldsymbol{x}_0) &= \psi(\chi(\boldsymbol{x}_0^A), \boldsymbol{x}_0^{\bar{A}}) = \psi(0, \boldsymbol{x}_0^{\bar{A}}) \\ &= \psi(\chi(\boldsymbol{0}^A), \boldsymbol{x}_0^{\bar{A}}) = \phi(\boldsymbol{0}^A, \boldsymbol{x}_0^{\bar{A}}).\end{aligned}$$



Exercise 3.9

Find all the modules of the following structure function:

$$\phi(\boldsymbol{x}) = (x_1 \cdot (x_2 \amalg x_3)) \amalg (x_4 \amalg x_5).$$

SOLUTION: (Only non-trivial modules are included here)

- $A_1 = \{2,3\}, \quad \chi_1(\boldsymbol{x}^{A_1}) = x_2 \amalg x_3, \quad \psi_1(\chi_1, \boldsymbol{x}^{\bar{A}_1}) = (x_1 \cdot \chi_1) \amalg (x_4 \amalg x_5)$
- $A_2 = \{1,2,3\}, \quad \chi_2(\boldsymbol{x}^{A_2}) = x_1(x_2 \amalg x_3), \quad \psi_2(\chi_2, \boldsymbol{x}^{\bar{A}_2}) = \chi_2 \amalg (x_4 \amalg x_5)$
- $A_3 = \{4,5\}, \quad \chi_3(\boldsymbol{x}^{A_3}) = x_4 \amalg x_5, \quad \psi_3(\chi_3, \boldsymbol{x}^{\bar{A}_3}) = (x_1(x_2 \amalg x_3)) \amalg \chi_3$
- $A_4 = \{1,2,3,4\}, \quad \chi_4(\boldsymbol{x}^{A_4}) = (x_1(x_2 \amalg x_3)) \amalg x_4, \quad \psi_4(\chi_4, \boldsymbol{x}^{\bar{A}_4}) = \chi_4 \amalg x_5$
- $A_5 = \{1,2,3,5\}, \quad \chi_5(\boldsymbol{x}^{A_5}) = (x_1(x_2 \amalg x_3)) \amalg x_5, \quad \psi_5(\chi_5, \boldsymbol{x}^{\bar{A}_5}) = \chi_5 \amalg x_4$



Exercise 3.11

Consider the system shown in the figure below.

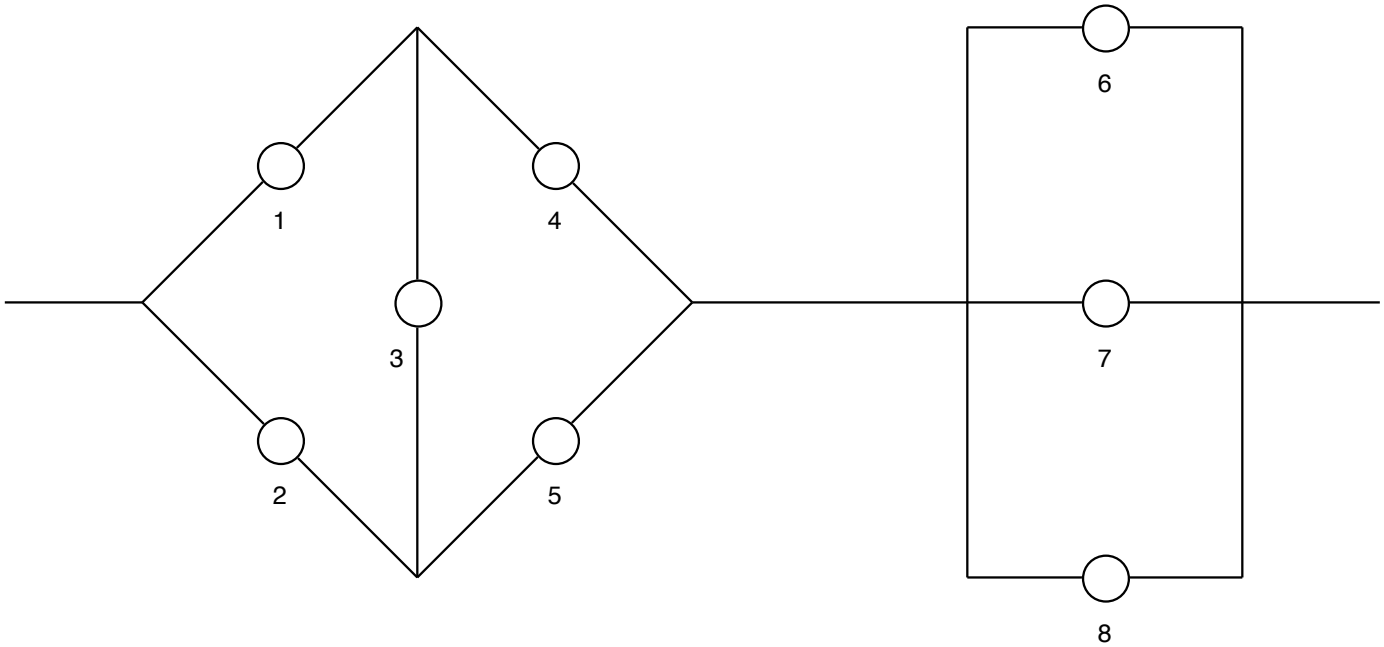


Figure: A system suited for modular decomposition.



Exercise 3.11 (cont.)

a) Find the structure function of the system by dividing the system into modules.

SOLUTION: We introduce the modules (A_1, χ_1) and (A_2, χ_2) where:

$$A_1 = \{1, 2, 3, 4, 5\}$$

$$\chi_1(\mathbf{x}^{A_1}) = x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))$$

$$A_2 = \{6, 7, 8\}$$

$$\chi_2(\mathbf{x}^{A_2}) = x_6 \amalg x_7 \amalg x_8$$

Since the two modules are connected in series, we get that:

$$\begin{aligned}\phi(\mathbf{x}) &= \chi_1(\mathbf{x}^{A_1}) \cdot \chi_2(\mathbf{x}^{A_2}) \\ &= [x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))] \cdot [x_6 \amalg x_7 \amalg x_8]\end{aligned}$$



Exercise 3.11 (cont.)

b) Find the structure function of the system without dividing the system into modules.

SOLUTION: There are several ways of doing this. Here, we choose to pivot w.r.t. component 3 (i.e., the brigde):

$$\phi(\mathbf{x}) = x_3\phi(1_3, \mathbf{x}) + (1 - x_3)\phi(0_3, \mathbf{x})$$

Case 1. Component 3 is *functioning*.
In this case the first system becomes a series of two parallel systems consisting of $1, 2$ and $4, 5$. This is again in series with the parallel connection of $6, 7, 8$. Hence we get:

$$\phi(1_3, \mathbf{x}) = (x_1 \amalg x_2) \cdot (x_4 \amalg x_5) \cdot (x_6 \amalg x_7 \amalg x_8)$$



Exercise 3.11 (cont.)

Case 2. Component 3 is *not functioning*.
In this case the first system becomes a parallel connection of the series connection of $1, 4$ and the series connection of $2, 5$. This is again is series with the parallel connection of $6, 7, 8$. Hence we get:

$$\phi(0_3, \mathbf{x}) = \big((x_1 x_4) \amalg (x_2 x_5)\big) \cdot (x_6 \amalg x_7 \amalg x_8)$$

Combining the two cases we get:

$$\begin{aligned}\phi(\mathbf{x}) &= x_3(x_1 \amalg x_2) \cdot (x_4 \amalg x_5) \cdot (x_6 \amalg x_7 \amalg x_8) \\ &\quad + (1 - x_3)\big((x_1 x_4) \amalg (x_2 x_5)\big) \cdot (x_6 \amalg x_7 \amalg x_8).\end{aligned}$$



Exercise 3.11 (cont.)

c) Compare the computational efforts in a) and b).

SOLUTION: In a) $\phi(\mathbf{x})$ is expressed as:

$$\phi(\mathbf{x}) = [x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))] \cdot [x_6 \amalg x_7 \amalg x_8]$$

In b) $\phi(\mathbf{x})$ is expressed as:

$$\begin{aligned}\phi(\mathbf{x}) = {} & x_3(x_1 \amalg x_2) \cdot (x_4 \amalg x_5) \cdot (x_6 \amalg x_7 \amalg x_8) \\ & + (1 - x_3)\big((x_1 x_4) \amalg (x_2 x_5)\big) \cdot (x_6 \amalg x_7 \amalg x_8).\end{aligned}$$

The formula obtained via the modular decomposition is easier to compute than the formula obtained with a direct approach since with the latter approach we need to compute $(x_6 \amalg x_7 \amalg x_8)$ twice.



Chapter 4

Exact computation of reliability of binary monotone systems



Computational complexity

Let:

$$n = \text{The size of the problem (e.g., number of components)}$$
$$t(n) = \text{The worst case running time of the algorithm as a function of } n$$
$$f(n) = \text{Some known non-negative increasing function of } n$$

The order of the algorithm is said to be $O(f(n))$ if and only if there exists a positive constant M and a positive integer n_0 such that:

$$t(n) \leq Mf(n), \text{ for all } n \geq n_0.$$

If f is a polynomial in n, we say that the algorithm is a polynomial time algorithm, while if f is an exponential function of n, we say that the algorithm is an exponential time algorithm.



Computational complexity (cont.)

- NP (for nondeterministic polynomial time) is a complexity class used to describe certain types of problems.
- NP contains many important problems, the hardest of which are called NP-complete problems.
- Open question: Is it possible to find a polynomial time algorithm for solving NP-complete problems. Conjecture: NO.
- The class of NP-hard problems is a class of problems that are, informally, at least as hard as the hardest problems in NP.
- The problem of computing the reliability of a binary monotone system is known to be NP-hard in the general case.



State space enumeration



State space enumeration

We know that the reliability of a binary monotone system (C, ϕ) is simply the expected value of $\phi(\boldsymbol{X})$.

Thus, by standard probability theory this can be calculated as:

$$h = E[\phi(\boldsymbol{X})] = \sum_{\boldsymbol{x} \in \{0,1\}^n} \phi(\boldsymbol{x}) P(\boldsymbol{X} = \boldsymbol{x})$$

NOTE:

- Calculating reliability using state space enumeration implies summing 2^n terms.
- If the components are dependent, the entire joint distribution of $\boldsymbol{X}$ is needed.



State space enumeration (cont.)

If the component reliabilities are $p_1, \ldots, p_n$, we may write:

$$P(X_i = x_i) = p_i^{x_i}(1 - p_i)^{1-x_i}, \quad x_i = 0, 1, \quad i = 1, \ldots, n.$$

Thus, if the components are independent, we have:

$$P(\boldsymbol{X} = \boldsymbol{x}) = \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}$$

Inserting this into the state space enumeration formula we get:

$$h(\boldsymbol{p}) = \sum_{\boldsymbol{x} \in \{0,1\}^n} \phi(\boldsymbol{x}) \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}.$$



State space enumeration (cont.)

NOTE: Due to the large number of terms in the expansion, the algorithm is at least of order $O(2^n)$. The order is typically greater since the work of computing $\phi(\cdot)$ for each $\boldsymbol{x}$ must be included as well.

It is possible to improve the algorithm by utilising that ϕ is non-decreasing in each argument:

If e.g., $\phi(\boldsymbol{x}_0) = 0$, it follows that $\phi(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} < \boldsymbol{x}_0$ as well. Hence, any state vector such that $\boldsymbol{x} < \boldsymbol{x}_0$ can be eliminated from the sum.



State space enumeration (cont.)

Let $C = \{1, 2, 3\}$, and let:

$$\phi(\boldsymbol{x}) = (x_1 \amalg x_2) \cdot x_3.$$

The component state variables are independent, and the component reliabilities are $P(X_i = 1) = p_i$, $i = 1, 2, 3$.

We then have:

$$\begin{aligned} h(\boldsymbol{p}) &= \phi(0,0,0) \cdot (1-p_1)(1-p_2)(1-p_3) \\ &\quad + \phi(1,0,0) \cdot p_1(1-p_2)(1-p_3) + \cdots + \phi(1,1,1) \cdot p_1 p_2 p_3 \\ &= p_1(1-p_2)p_3 + (1-p_1)p_2 p_3 + p_1 p_2 p_3. \end{aligned}$$



State space enumeration (cont.)

NOTE: There are exactly 3 vectors such that $\phi(\boldsymbol{x}) = 1$:

$$\boldsymbol{x}_1 = (1, 0, 1), \quad \boldsymbol{x}_2 = (0, 1, 1), \quad \boldsymbol{x}_3 = (1, 1, 1).$$

Hence, the reliability function contains just 3 terms, not $2^3 = 8$ terms.

NOTE: All terms have p_3 as a common factor, and:

$$p_1(1 - p_2) + (1 - p_1)p_2 + p_1 p_2 = p_1 \amalg p_2$$

Hence, $h(\boldsymbol{p})$ can be simplified to:

$$\begin{aligned} h(\boldsymbol{p}) &= [p_1(1 - p_2) + (1 - p_1)p_2 + p_1 p_2] \cdot p_3 \\ &= (p_1 \amalg p_2)p_3 \end{aligned}$$

The last expression can be obtained directly from the structure function since the system is an s-p-system.

