STK3405 – Lecture 6

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Section 4.2.

The multiplication method



The multiplication method

Consider a binary monotone system with minimal path sets:

$$P_1, \ldots, P_p,$$

and minimal cut sets:

$$K_1, \ldots, K_k.$$

We then have:

$$\phi(\boldsymbol{X}) = \coprod_{j=1}^{p} \prod_{i \in P_j} X_i = 1 - [\prod_{j=1}^{p} (1 - \prod_{i \in P_j} X_i)]$$

$$= \prod_{j=1}^{k} \coprod_{i \in K_j} X_i = \prod_{j=1}^{k} [1 - \prod_{i \in K_j} (1 - X_i)].$$



The multiplication method (cont.)

By expanding either the formula based on the minimal path sets, or the formula based on the minimal cut sets, and using that $X_i^r = X_i$, $i = 1, \ldots, n$, $r = 1, 2, \ldots$, we eventually get an expression of the form:

$$\phi(\boldsymbol{X}) = \sum_{S \subseteq C} \delta(S) \prod_{i \in S} X_i$$

where for all $S \subseteq C$, $\delta(S)$ denotes the coefficient of the term associated with the product $\prod_{i \in S} X_i$.

The δ-function is called the signed domination function of the system.



The multiplication method (cont.)

By taking the expectation on both sides, and assuming that the component state variables are independent, we obtain:

$$h(\boldsymbol{p}) = E[\phi(\boldsymbol{X})] = \sum_{S \subseteq C} \delta(S) \prod_{i \in S} E[X_i] = \sum_{S \subseteq C} \delta(S) \prod_{i \in S} p_i$$

The sum contains 2^n terms.

Hence, the multiplication method also has at least order $O(2^n)$.

Note, however, that lots of terms in this formula vanish since typically $\delta(S) = 0$ for many of the subsets S of the component set C.

Improved versions of this method utilizes this in order to speed up the calculations.



The multiplication method (cont.)

We now introduce the following version of the structure function of a binary monotone system (C, ϕ):

$$\phi(B) = \phi(\mathbf{1}^B, \mathbf{0}^{\bar{B}}), \text{ for all } B \subseteq C.$$

Thus, $\phi(B)$ represents the state of the system given that all the components in the set B are functioning, while all the components in the set $\bar{B} = C \setminus B$ are failed.

Note that if B is the set of functioning components in the system, then:

$$\prod_{i \in S} X_i = \begin{cases} 1 & \text{if } S \subseteq B \\ 0 & \text{otherwise} \end{cases}$$

Hence, we get the following simplified expression:

$$\phi(B) = \sum_{S \subseteq C} \delta(S) \prod_{i \in S} X_i = \sum_{S \subseteq B} \delta(S) \tag{1}$$



The signed domination formula

We now have the following important result:

Theorem (Signed domination formula)

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B), \text{ for all } A \subseteq C.$$

NOTE: This formula allows us (at least in principle) to compute the signed domination function without using the minimal path and cut sets.

For certain classes of systems this can be used to establish formulas allowing much faster calculations.



In order to prove the theorem we need the following lemma:

Lemma

Let A and S be two subsets of a given set C. Then we have:

$$\sum_{S \subseteq B \subseteq A} (-1)^{|A|-|B|} = \begin{cases} 1 & \text{if } S = A \\ 0 & \text{otherwise} \end{cases} = \mathrm{I}(S = A)$$

PROOF: We consider three possible cases:

CASE 1. $S \not\subseteq A$. In this case there exists no set B such that $S \subseteq B \subseteq A$. Hence, obviously the sum is equal to zero.

CASE 2. $S = A$. In this case the sum contains only one term, and we get:

$$\sum_{S \subseteq B \subseteq A} (-1)^{|A|-|B|} = (-1)^{|A|-|A|} = 1$$



CASE 3. $S \subset A$. Since in this case S is a proper subset of A, there exists at least one element $i \in A \setminus S$.

We then split the sum into two parts as follows:

$$\sum_{S \subseteq B \subseteq A} (-1)^{|A|-|B|} = \sum_{S \subseteq B \subseteq A,\, i \notin B} (-1)^{|A|-|B|} + \sum_{S \subseteq B \subseteq A,\, i \in B} (-1)^{|A|-|B|}$$

$$= \sum_{S \subseteq B \subseteq A,\, i \notin B} (-1)^{|A|-|B|} + \sum_{S \subseteq B \subseteq A,\, i \notin B} (-1)^{|A|-|B \cup i|}$$

$$= \sum_{S \subseteq B \subseteq A,\, i \notin B} (-1)^{|A|-|B|} - \sum_{S \subseteq B \subseteq A,\, i \notin B} (-1)^{|A|-|B|} = 0$$



Proof of the sign domination formula

We can now prove the signed domination formula, and start out by recalling that:

$$\phi(B) = \sum_{S \subseteq B} \delta(S),$$

Hence, by using the lemma, we get:

$$\begin{aligned}
\sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B) &= \sum_{B \subseteq A} (-1)^{|A|-|B|} \sum_{S \subseteq B} \delta(S) \\
&= \sum_{S \subseteq A} \delta(S) \sum_{S \subseteq B \subseteq A} (-1)^{|A|-|B|} \\
&= \sum_{S \subseteq A} \delta(S)\, \mathsf{I}(S = A) = \delta(A)
\end{aligned}$$

as claimed.



Properties of the signed domination

Theorem

Let (C, ϕ) be a binary monotone system, and let $A \subseteq C$.

If $\phi(A) = 0$, then $\delta(A) = 0$ as well.

PROOF: Since ϕ is nondecreasing in each argument, it follows from the assumption that $\phi(B) = 0$ for all $B \subseteq A$. Hence,

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B) = 0.$$



Properties of the signed domination (cont.)

Theorem

Let (C, ϕ) be a binary monotone system, let $A \subseteq C$, and let $i \in A$.

If $\phi(B \cup i) = \phi(B)$ for all $B \subseteq A \setminus i$, then $\delta(A) = 0$.

PROOF:

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B)$$

$$= \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B|} \phi(B) + \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B \cup i|} \phi(B \cup i)$$

$$= \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B|} \phi(B) - \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B|} \phi(B) = 0$$



Properties of the signed domination (cont.)

Theorem

Let (C, ϕ) be a binary monotone system, let $P \subseteq C$ be a minimal path set.

Then $\delta(P) = 1$.

PROOF: Since P is a minimal path set, it follows that $\phi(P) = 1$ and that $\phi(B) = 0$ for all $B \subset P$. Hence,

$$\delta(P) = \sum_{B \subseteq P} (-1)^{|P|-|B|} \phi(B)$$

$$= (-1)^{|P|-|P|} \phi(P) = 1.$$



The multiplication method (cont.)

Consider a binary monotone system (C, ϕ) where $C = \{1, 2, 3\}$, with minimal path sets $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$. Using the multiplication method we obtain:

$$\begin{aligned}\phi(\boldsymbol{X}) &= (X_1 X_2) \amalg (X_1 X_3) = 1 - (1 - X_1 X_2)(1 - X_1 X_3) \\ &= 1 - (1 - X_1 X_2 - X_1 X_3 + X_1^2 X_2 X_3) \\ &= X_1 X_2 + X_1 X_3 - X_1 X_2 X_3,\end{aligned}$$

where we have used that $X_1^2 = X_1$.

Thus, $\delta(\{1, 2\}) = \delta(\{1, 3\}) = 1$, $\delta(\{1, 2, 3\}) = -1$, while $\delta(A) = 0$ for all other subsets of C.



The multiplication method (cont.)

Note that these coefficients can be obtained using the signed domination formula and the theorems as well since:

$$\delta(\{1,2\}) = 1, \quad \text{since } \{1,2\} \text{ is a minimal path set}$$

$$\delta(\{1,3\}) = 1, \quad \text{since } \{1,3\} \text{ is a minimal path set}$$

$$\delta(\{1,2,3\}) = (-1)^{|\{1,2,3\}|-|\{1,2\}|}\phi(\{1,2\}) + (-1)^{|\{1,2,3\}|-|\{1,3\}|}\phi(\{1,3\})$$

$$+ (-1)^{|\{1,2,3\}|-|\{1,2,3\}|}\phi(\{1,2,3\}) = -1 - 1 + 1 = -1.$$

Having derived the formula for the structure function ϕ, we immediately obtain the reliability function:

$$h(\boldsymbol{p}) = p_1 p_2 + p_1 p_3 - p_1 p_2 p_3.$$



Section 4.3.

The inclusion-exclusion method



The inclusion-exclusion method

Consider a binary monotone system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$. We then introduce the events:

$$E_j = \{\text{All of the components in } P_j \text{ are functioning}\}, \quad j = 1, \ldots, p.$$

Since the system is functioning if and only if at least one of the minimal path sets is functioning, we have:

$$\phi = 1 \quad \text{if and only if} \quad \bigcup_{j=1}^{p} E_j \text{ holds true.}$$



The inclusion-exclusion method (cont.)

We then use the inclusion-exclusion formula and get:

$$
\begin{aligned}
h &= P(\bigcup_{j=1}^{p} E_j) \\
&= P(E_1) + P(E_2) + \cdots + P(E_p) \\
&- P(E_1 \cap E_2) - P(E_1 \cap E_3) - \cdots - P(E_{p-1} \cap E_p) \qquad \text{(All event pairs)} \\
&+ P(E_1 \cap E_2 \cap E_3) + \cdots + P(E_{p-2} \cap E_{p-1} \cap E_p) \qquad \text{(All event triplets)} \\
&\cdots \\
&+ (-1)^{p-1} P(E_1 \cap \cdots \cap E_p).
\end{aligned}
$$

NOTE: The number of terms is $2^p - 1$. However, typically many of the terms can be merged as they correspond to the same component set.



The inclusion-exclusion method (cont.)

NOTE: An event of form $E_{i_1} \cap \cdots \cap E_{i_r}$ occurs if and only if all the components in the set $P_{i_1} \cup \cdots \cup P_{i_r}$ are functioning. When the component state variables are independent, we get:

$$P(E_{i_1} \cap \cdots \cap E_{i_r}) = \prod_{i \in P_{i_1} \cup \cdots \cup P_{i_r}} p_i.$$



The inclusion-exclusion method (cont.)

Consider a binary monotone system (C, ϕ) where $C = \{1, 2, 3, 4\}$ with minimal path sets $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$, $P_3 = \{2, 3, 4\}$.

Assuming that the component state variables are independent, we then get:

$$P(E_1) = p_1 p_2, \quad P(E_2) = p_1 p_3, \quad P(E_3) = p_2 p_3 p_4$$
$$P(E_1 \cap E_2) = p_1 p_2 p_3, \quad P(E_1 \cap E_3) = P(E_2 \cap E_3) = p_1 p_2 p_3 p_4$$
$$P(E_1 \cap E_2 \cap E_3) = p_1 p_2 p_3 p_4.$$

Hence, the reliability of the system is:

$$\begin{aligned} h(\boldsymbol{p}) &= p_1 p_2 + p_1 p_3 + p_2 p_3 p_4 - p_1 p_2 p_3 - 2 p_1 p_2 p_3 p_4 + p_1 p_2 p_3 p_4 \\ &= p_1 p_2 + p_1 p_3 + p_2 p_3 p_4 - p_1 p_2 p_3 - p_1 p_2 p_3 p_4. \end{aligned}$$



The inclusion-exclusion method (cont.)

NOTE: The final expression for the reliability function is exactly the same as we get using the multiplication methods:

$$h(\boldsymbol{p}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} p_i,$$

where δ denotes the signed domination function. However, the steps we take in order to get this expression is different.

By using the inclusion-exclusion formula we see that all terms in the expansion correspond to sets $A \subseteq C$ which are unions of minimal path sets.

If a set $A \subseteq C$ is *not* a union of minimal path sets, it follows that $\delta(A) = 0$.



The inclusion-exclusion method (cont.)

If $P_{i_1}, \ldots, P_{i_r}$ is a collection of minimal path sets such that:

$$\bigcup_{j=1}^{r} P_{i_j} = A,$$

the collection is said to be a formation of the set A.

The formation is odd if r is odd, and even if r is even.

An odd formation contributes with a coefficient $+1$, while an even formation contributes with a coefficient -1.



The inclusion-exclusion method (cont.)

Simplifying the expansion, all terms corresponding to formations of the same set are merged. Hence, it follows that we have:

Theorem

Let (C, ϕ) be a binary monotone system with minimal path sets $P_1, \ldots, P_p$, and let δ denote the signed domination function of the system. Then for all $A \subseteq C$ we have:

$$\begin{aligned} \delta(A) = & \text{ The number of odd formations of } A \\ & - \text{ The number of even formations of } A. \end{aligned}$$

In particular $\delta(A) = 0$ if A is not a union of minimal path sets.

As a corollary we obtain the following result:

Corollary

If (C, ϕ) is a binary monotone system which is not coherent, then $\delta(C) = 0$.

An upper bound on the system reliability

Skipping all higher order terms and keeping the probabilities of the individual events $E_1, \ldots, E_p$ only, we get an upper bound on the system reliability:

$$h \leq \sum_{j=1}^{p} P(E_j).$$

Given that p is moderate, and that we are given the minimal path sets, this upper bound is easy to calculate.



An lower bound on the system reliability

We then consider the minimal cut sets of the system $K_1, \ldots, K_k$, and introduce:

$$F_j = \{\text{All the components in } K_j \text{ are failed}\}, \quad j = 1, \ldots, k.$$

Since the system is failed if and only if all the components in at least one cut set are failed, we have:

$$1 - h = P(\bigcup_{j=1}^{k} F_j).$$

An upper bound on $1 - h$ is then given by:

$$1 - h \leq \sum_{j=1}^{k} P(F_j).$$

Combining the upper and lower bounds, we get:

$$1 - \sum_{j=1}^{k} P(F_j) \leq h \leq \sum_{j=1}^{p} P(E_j).$$



Bounds on the system reliability (cont.)

If the components are independent, we get:

$$1 - \sum_{j=1}^{k} \prod_{i \in K_j} (1 - p_i) \leq h(\boldsymbol{p}) \leq \sum_{j=1}^{p} \prod_{i \in P_j} p_i.$$

If the component reliabilites are close to 1, the lower bound turns out to be very good, while the upper bound will be crude and possibly greater than 1.

If the component reliabilites are close to 0, the lower bound will be crude and possibly less than 0, while the upper bound turns out to be very good.



Bounds on the system reliability (cont.)

In order to avoid bounds outside of the interval $[0, 1]$, one would typically replace the bounds by:

$$\max(1 - \sum_{j=1}^{k} \prod_{i \in K_j} (1 - p_i),\ 0) \leq h(\boldsymbol{p}) \leq \min(\sum_{j=1}^{p} \prod_{i \in P_j} p_i,\ 1).$$

