UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK3405/4405 — Introduction to risk and reliability analysis

Day of examination: Friday December 8, 2017

Examination hours: 09.00 – 13.00

This problem set consists of 4 pages.

Appendices: None

Permitted aids: Calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

In this problem we consider measures of reliability importance. Let (C, ϕ) be a binary monotone system with component set $C = \{1, \dots, n\}$ and structure function ϕ. The Birnbaum measure of the reliability importance of component $i \in C$ at time $t \geq 0$ is defined as:

$$\begin{aligned} I_B^{(i)}(t) &= \mathrm{P}(\text{Component } i \text{ is critical for the system at time } t) \\ &= \mathrm{P}(\phi(1_i \boldsymbol{X}(t)) - \phi(0_i \boldsymbol{X}(t)) = 1) \\ &= \mathrm{E}[\phi(1_i \boldsymbol{X}(t)) - \phi(0_i \boldsymbol{X}(t))], \end{aligned}$$

where $\boldsymbol{X}(t) = (X_1(t), \dots, X_n(t))$ denotes the vector of component state variables at time $t \geq 0$. We assume that $\mathrm{P}(X_i(t) = 1) = p_i(t)$, $i = 1, \dots, n$, and let $\boldsymbol{p}(t) = (p_1(t), \dots, p_n(t))$ denote the vector of component reliabilities at time $t \geq 0$. We let $h(t) = \mathrm{P}(\phi(\boldsymbol{X}(t)) = 1) = \mathrm{E}[\phi(\boldsymbol{X}(t)]$ denote the reliability of the system at time $t \geq 0$. If $X_1(t), \dots, X_n(t)$ are stochastically independent, we may write $h(t) = h(\boldsymbol{p}(t))$.

(a) Let T_S denote the lifetime of the system, and let T_i denote the lifetime of component i, $i = 1, \dots n$. Explain briefly that for $t \geq 0$ we have $T_S > t$ if and only if $\phi(\boldsymbol{X}(t)) = 1$, and use this to show that:

$$I_B^{(i)}(t) = \mathrm{P}(T_S > t | T_i > t) - \mathrm{P}(T_S > t | T_i \leq t), \quad t \geq 0, \quad i = 1, \dots, n.$$

(b) Assume that $X_1(t), \dots, X_n(t)$ are stochastically independent. Show that we then have:

$$I_B^{(i)}(t) = \frac{\partial h(\boldsymbol{p}(t))}{\partial p_i(t)}, \quad t \geq 0, \quad i = 1, \dots, n.$$

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We now introduce the Birnbaum measure of the *joint* reliability importance of the components $i, j \in C$ at time $t \geq 0$ defined by:

$$I_B^{(i,j)}(t) = \mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(1_i, 0_j \boldsymbol{X}(t)) \\ - \phi(0_i, 1_j \boldsymbol{X}(t)) + \phi(0_i, 0_j \boldsymbol{X}(t))].$$

(c) Explain briefly if $I_B^{(i,j)}(t) > 0$, this implies that:

$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 1_j \boldsymbol{X}(t))] > \mathrm{E}[\phi(1_i, 0_j \boldsymbol{X}(t)) - \phi(0_i, 0_j \boldsymbol{X}(t))],$$
$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(1_i, 0_j \boldsymbol{X}(t))] > \mathrm{E}[\phi(0_i, 1_j \boldsymbol{X}(t)) - \phi(0_i, 0_j \boldsymbol{X}(t))],$$

while the opposite inequalities hold if $I_B^{(i,j)}(t) < 0$. Use this to give a practical interpretation of the sign of $I_B^{(i,j)}(t)$.

(d) Show that for $i, j \in C$ and $t \geq 0$ we have:

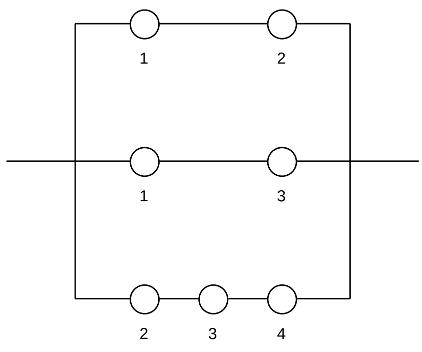
$$I_B^{(i,j)}(t) = \mathrm{P}(T_S > t | T_i > t, T_j > t) - \mathrm{P}(T_S > t | T_i > t, T_j \leq t) \\ - \mathrm{P}(T_S > t | T_i \leq t, T_j > t) + \mathrm{P}(T_S > t | T_i \leq t, T_j \leq t).$$

(e) Assume that $X_1(t), \ldots, X_n(t)$ are stochastically independent. Show that we then have:

$$I_B^{(i,j)}(t) = \frac{\partial^2 h(\boldsymbol{p}(t))}{\partial p_i(t) \partial p_j(t)}, \quad t \geq 0, \quad i, j = 1, \ldots, n.$$

(f) We still assume that $X_1(t), \ldots, X_n(t)$ are stochastically independent, and let $i, j \in C$ and $t \geq 0$. Moreover, assume that $0 < p_i(t) < 1$ for all $i \in C$ and that $n \geq 3$. Show that $I_B^{(i,j)}(t) > 0$ if (C, ϕ) is a series system, while $I_B^{(i,j)}(t) < 0$ if (C, ϕ) is a parallel system. Give a brief comment to this result.

Problem 2



In this problem we consider a binary monotone system (C, ϕ). The system is shown in the block diagram in the figure above. The component set of the system is $C = \{1, 2, 3, 4\}$.

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We let $\boldsymbol{X} = (X_1, X_2, X_3, X_4)$ denote the vector of component state variables and assume throughout this problem that X_1, X_2, X_3, X_4 are stochastically independent. Moreover, we let $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$ denote the vector of component reliabilities where $p_i = P(X_i = 1)$, $i = 1, 2, 3, 4$. We assume that $0 < p_i < 1$, $i = 1, 2, 3, 4$.

(a) Find the minimal path and cut sets of (C, ϕ).

(b) Show that the structure function of the system can be expressed as:

$$\begin{aligned}\phi(\boldsymbol{X}) &= X_4[X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3] \\ &\quad + (1 - X_4)[X_1X_2 + X_1X_3 - X_1X_2X_3],\end{aligned}$$

and use this to find the reliability of the system, $h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})]$.

You may use that the Birnbaum measure of the reliability importance of component $i \in C$ is given by:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \quad i = 1, 2, 3, 4,$$

and that the Birnbaum measure of the joint reliability importance of the components $i, j \in C$ is given by:

$$I_B^{(i,j)} = \frac{\partial^2 h(\boldsymbol{p})}{\partial p_i \partial p_j}, \quad i, j = 1, 2, 3, 4.$$

(c) Show that:

$$I_B^{(4)} = p_2p_3 - p_1p_2p_3.$$

(d) Show that:

$$I_B^{(1,4)} < 0, \text{ og } I_B^{(i,4)} > 0, \quad i = 2, 3.$$

Give a brief comment to these results.

Problem 3

If $X_1, X_2, \ldots$ is an infinite sequence of independent identically distributed stochastic variables where $\mathrm{E}[X_i] = \mu < \infty$, it can be shown that:

$$P(\bar{X}_n \to \mu) = 1,$$

der $\bar{X}_n = (X_1 + \cdots X_n)/n$, $n = 1, 2, \ldots$.

Let $\{S(t)\}$ be a stochastic process where $S(t)$ denotes the state of the process at time $t \geq 0$. We say that $\{S(t)\}$ is a *pure jump process* if $S(t)$ can be expressed as:

$$S(t) = S(0) + \sum_{j=1}^{\infty} \mathrm{I}(T_j \leq t) J_j, \qquad t \geq 0,$$

where $0 = T_0 < T_1 < T_2 < \cdots$ is a sequence of stochastic points of time, and $J_1, J_2, \ldots$ is a sequence of stochastic *jumps*.

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We introduce:

$$N(t) = \sum_{j=1}^{\infty} I(T_j \leq t) = \text{The number of jumps in [0, t]}.$$

The process $\{S(t)\}$ is said to be *regular* if $P(N(t) < \infty) = 1$ for all $t > 0$.

We then let $\Delta_j = T_j - T_{j-1}$, $j = 1, 2, \ldots$.

(a) Show that if the sequence $\{\Delta_j\}$ contains an infinite subsequence, $\{\Delta_{k_j}\}$, of independent, identically distributed stochastic variables such that $E[\Delta_{k_j}] = d > 0$, then $\{S(t)\}$ us regular.

(b) Explain why regularity is important for simulations of pure jump processes.