UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK3405/STK4405 — Elementary introduction to risk and reliability analysis.
Day of examination:	Wednesday 19. December 2018.
Examination hours:	14.30 – 18.30.
This problem set consists of 4 pages.	
Appendices:	None.
Permitted aids:	Accepted calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subpoints will be equally weighted in the marking.

Problem 1

Consider the binary monotone system (C, ϕ) shown in Figure 1. The component set of the system is $C = \{1, 2, \dots, 6\}$. Let $\boldsymbol{X} = (X_1, X_2, \dots, X_6)$ denote the vector of component state variables, and assume throughout this problem that $X_1, X_2, \dots, X_6$ are stochastically independent. Let $\boldsymbol{p} = (p_1, p_2, \dots, p_6)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, \dots, 6$. We assume that $0 < p_i < 1$ for $i = 1, 2, \dots, 6$.

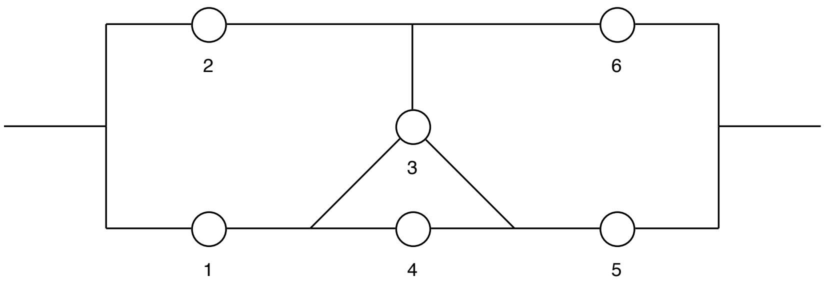

Figure 1: A binary monotone system of 6 components.

a) Find the minimal path and cut sets of the system.

b) Use the result in a) to find an expression for the structure function of

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the system, and explain briefly how this can be used to find the system reliability. A detailed calculation is not required.

c) Use the factoring algorithm to derive the reliability of the system in a different way from the one in b).

d) What is the definition of the Birnbaum measure for the reliability importance of a component?

e) What is the reliability importance of component 3 according to the Birnbaum measure? How can you use this result to find the structural importance of component 3?

f) Assume that $p_i = p$ for $i = 1, 2, \ldots, 6$, i.e., that all the components have the same component reliability. What can you say about the reliability importance of the other 5 components?

Problem 2

Consider a binary monotone system (C, ϕ), where $C = \{1, 2, 3\}$ and where the structure function ϕ is given by:

$$\phi(\boldsymbol{X}) = \mathrm{I}(\sum_{i=1}^{3} X_i \geq 2).$$

Here $\boldsymbol{X} = (X_1, X_2, X_3)$ denotes the vector of component state variables and $\mathrm{I}(\cdot)$ denotes the indicator function.

a) Show that the structure function ϕ can be written as:

$$\phi(\boldsymbol{X}) = X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3.$$

In the following we assume that:

$$X_i = Y_0 \cdot Y_i, \quad i = 1, 2, 3,$$

where Y_0, Y_1, Y_2, Y_3 are independent binary stochastic variables and:

$$P(Y_0 = 1) = \theta,\ P(Y_1 = 1) = P(Y_2 = 1) = P(Y_3 = 1) = q,$$

where $0 < \theta < 1$ and $0 < q < 1$.

b) Explain why this implies that X_1, X_2, X_3 are associated stochastic variables.

We then introduce $h = \mathrm{E}[\phi(\boldsymbol{X})] = P(\phi(\boldsymbol{X}) = 1)$.

c) Show that:

$$h = h(\theta, q) = \theta q^2(3 - 2q).$$

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Assume that we ignore the dependence between the X_is, and instead computes the system reliability as if X_1, X_2, X_3 are independent and:

$$P(X_i = 1) = \theta q, \quad i = 1, 2, 3.$$

Let $\tilde{h}$ denote the system reliability we then get.

d) Show that:

$$\tilde{h} = \tilde{h}(\theta, q) = \theta^2 q^2 (3 - 2\theta q).$$

e) Assume that $\theta = \frac{1}{2}$. Show that we then have $\tilde{h} < h$ for all $0 < q < 1$.

f) Assume instead that $\theta = \frac{3}{4}$. What can you say about the relationship between $\tilde{h}$ and h in this case?

Problem 3

Let (C, ϕ) be a binary monotone system, and let $\boldsymbol{X}$ denote the vector of component state variables. In this problem we consider how the system reliability $h = P(\phi(\boldsymbol{X}) = 1)$ can be estimated using Monte Carlo simulation. The simplest Monte Carlo estimate is:

$$\hat{h}_{MC} = \frac{1}{N} \sum_{r=1}^{N} \phi(\boldsymbol{X}_r),$$

where $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ are data generated from the distribution of $\boldsymbol{X}$.

In order to improve this estimate we let $S = S(\boldsymbol{X})$ be a stochastic variable with values in the set $\{s_1, \ldots, s_k\}$. We assume that the distribution of S is known, and introduce:

$$\theta_j = E[\phi | S = s_j], \quad j = 1, \ldots, k.$$

We then use Monte Carlo simulation in order to estimate $\theta_1, \ldots, \theta_k$, and generate data from the conditional distribution of $\boldsymbol{X}$ given S. We let $\{\boldsymbol{X}_{r,j} : r = 1, \ldots, N_j\}$ denote the vectors generated from the distribution of $\boldsymbol{X}$ given that $S = s_j$, $j = 1, \ldots, k$, and get the following estimates:

$$\hat{\theta}_j = \frac{1}{N_j} \sum_{r=1}^{N_j} \phi(\boldsymbol{X}_{r,j}), \quad j = 1, \ldots, k.$$

These estimates are then combined into the following estimate of the system reliability:

$$\hat{h}_{CMC} = \sum_{j=1}^{k} \hat{\theta}_j P(S = s_j).$$

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a) Show that $E[\hat{h}_{CMC}] = h$ and that the variance of the estimate is given by:

$$\mathrm{Var}(\hat{h}_{CMC}) = \sum_{j=1}^{k} \frac{1}{N_j} \mathrm{Var}(\phi | S = s_j)[P(S = s_j)]^2$$

b) Assume that $N_j \approx N \cdot P(S = s_j)$, $j = 1, \ldots, k$. Show that we then have:

$$\mathrm{Var}(\hat{h}_{CMC}) \approx \frac{1}{N}(\mathrm{Var}(\phi) - \mathrm{Var}[E(\phi | S)]),$$

and explain briefly why this implies that $\mathrm{Var}(\hat{h}_{CMC}) \leq \mathrm{Var}(\hat{h}_{MC})$.

c) What should one take into account when choosing S?

END