UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK3405/4405 — Introduction to risk and reliability analysis
Day of examination:	Monday November 25th 2019.
Examination hours:	14.30 – 18.30.
This problem set consists of 4 pages.	
Appendices:	None.
Permitted aids:	Approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subpoints will be equally weighted in the marking.

Problem 1

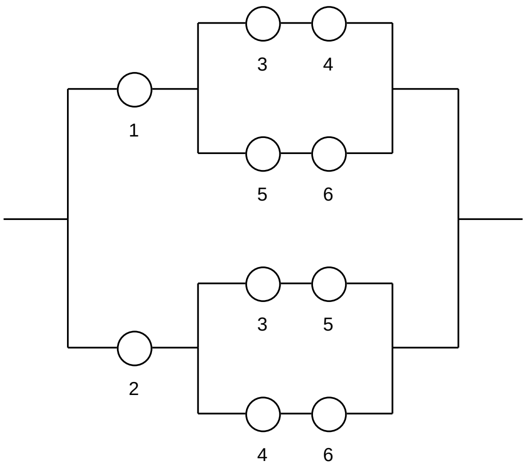

Figure 1: Block diagram of (C, ϕ)

Consider the binary monotone system (C, ϕ) shown in Figure 1. The component set of the system is $C = \{1, 2, \ldots, 6\}$. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_6)$ denote the vector of component state variables, and assume throughout this problem that $X_1, X_2, \ldots, X_6$ are stochastically independent. Let $\boldsymbol{p} = (p_1, p_2, \ldots, p_6)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, \ldots, 6$.

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a) Find the minimal path sets (4 sets) and the minimal cut sets (7 sets) of the system.

b) We let $h(\boldsymbol{p}) = P(\phi = 1)$ denote the reliability function of the system. Show that:

$$h(1_1, 1_2, \boldsymbol{p}) = E[\phi(1_1, 1_2, \boldsymbol{X})] = (p_3 \amalg p_6) \cdot (p_4 \amalg p_5),$$

$$h(1_1, 0_2, \boldsymbol{p}) = E[\phi(1_1, 0_2, \boldsymbol{X})] = (p_3 \cdot p_4) \amalg (p_5 \cdot p_6),$$

$$h(0_1, 1_2, \boldsymbol{p}) = E[\phi(0_1, 1_2, \boldsymbol{X})] = (p_3 \cdot p_5) \amalg (p_4 \cdot p_6),$$

and use this to find $h(\boldsymbol{p})$.

In the remaining part of this problem we assume that all components have equal reliability p, i.e., $p_1 = \cdots = p_6 = p$. The reliability function can then be written as $h(p)$ instead of $h(\boldsymbol{p})$.

c) Use the results from (b) to show that:

$$\begin{aligned} h(p) &= p^2 \cdot (2p - p^2)^2 + 2p(1-p) \cdot (2p^2 - p^4) \\ &= p^4 \cdot (2-p)^2 + 2p^3(1-p)(2-p^2) \end{aligned}$$

In particular, show that:

$$h(\frac{1}{2}) = 23 \cdot (\frac{1}{2})^6$$

d) Let $S = \sum_{i=1}^{6} X_i$. Explain why the distribution of S is given by:

$$P(S = s) = \binom{6}{s} p^s (1-p)^{6-s}, \quad s = 0, 1, \ldots, 6.$$

e) Show that:

$$h(p) = \sum_{s=0}^{6} b_s p^s (1-p)^{6-s}$$

where b_s denotes the number of path sets (minimal and non-minimal) having exactly s components, $s = 0, 1, \ldots, 6$.

f) Show that:

$$\sum_{s=0}^{6} b_s = 23.$$

g) Finally, determine $b_0, b_1, \ldots, b_6$.

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Problem 2

If $T_1, \ldots, T_n$ are random variables, and we let $\mathbf{T} = (T_1, \ldots, T_n)$, we say that $T_1, \ldots, T_n$ are *associated* if

$$\mathrm{Cov}(\Gamma(\mathbf{T}), \Delta(\mathbf{T})) \geq 0,$$

for all binary, non-decreasing functions Γ and Δ.

a) Prove that non-decreasing functions of associated random variables are associated.

b) Assume that $T_1, \ldots, T_n$ are associated random variables such that $0 \leq T_i \leq 1$, $i = 1, \ldots, n$. Prove that

$$E[\prod_{i=1}^{n} T_i] \geq \prod_{i=1}^{n} E[T_i] \quad \text{and} \tag{1}$$

$$E[\coprod_{i=1}^{n} T_i] \leq \coprod_{i=1}^{n} E[T_i]. \tag{2}$$

c) Interpret the inequalities (1) and (2) by applying them to the binary component state variables $X_1, \ldots, X_n$.

d) Let $X_1, \ldots, X_n$ be the associated component states of a binary monotone system (C, ϕ) with minimal path series structures $(P_1, \rho_1), \ldots, (P_p, \rho_p))$ and minimal cut parallel $(K_1, \kappa_1), \ldots, (K_k, \kappa_k))$. Prove that

$$\prod_{j=1}^{k} P(\kappa_j(\mathbf{X}^{K_j}) = 1) \leq h \leq \coprod_{j=1}^{p} P(\rho_j(\mathbf{X}^{P_j}) = 1). \tag{3}$$

Hint: Use the results from items a) and b).

e) Make the same assumptions as in item d), and assume in addition that the component states are independent with component reliabilities $p_1, p_2, \ldots p_n$. Use the result in d) to prove that

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\mathbf{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i. \tag{4}$$

f) Consider the system in Problem 1. Assume that all components have the same component reliability $p = 0.9$. Compute the bounds in inequality (4) and comment on how well they approximate the actual system reliability in this case.

(Continued on page 4.)

g) In points d) and e), you have found upper and lower bounds for the system reliability. In which cases is it particularly important to have such bounds?

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