Argument STK2520 in 2011, problem 1e).

Note the simple relationship between r_0 and x_0. We utilize that

$$E(X_k|x_0) = a^k x_0 \qquad \text{and} \qquad \text{sd}(X_k|x_0) = \sqrt{\frac{1-a^{2k}}{1-a^2}}\,\sigma$$

and that if $Y \sim N(\theta, \tau)$, then

$$E(e^Y) = e^{\theta+\tau^2/2} \qquad \text{and} \qquad \text{sd}(e^Y) = E(e^Y)\sqrt{e^{\tau^2}-1}.$$

With $Y = X_k|x_0$ so that $\theta = E(X_k|x_0)$ and $\tau = \text{sd}(X_k|x_0)$ the expression on the left yields

$$E(r_k|r_0) = E(\xi e^{-\frac{1}{2}\sigma^2/(1-a^2)+X_k}) = \xi e^{-\frac{1}{2}\sigma^2/(1-a^2)} E(e^{X_k}|x_0) = \xi e^{-\frac{1}{2}\sigma^2/(1-a^2)} e^{E(X_k|x_0)+\frac{1}{2}\text{var}(X_k|x_0)}$$

or when we insert for $E(X_k|x_0)$ and $\text{var}(X_k|x_0)$ this becomes[1]

$$E(r_k|r_0) = \xi e^{a^k x_0 - \sigma^2 a^{2k}/\{2(1-a^2)\}}$$

The conditional standard deviation is

$$\text{sd}(r_k|r_0) = \text{sd}(\xi e^{-\frac{1}{2}\sigma^2/(1-a^2)+X_k}|x_0) = \xi e^{-\frac{1}{2}\sigma^2/(1-a^2)}\text{sd}(e^{X_k}|x_0))$$

or

$$\text{sd}(r_k|r_0) = \xi e^{-\frac{1}{2}\sigma^2/(1-a^2)} E(e^{X_k}|x_0)\sqrt{e^{\text{var}(X_k|x_0)}-1}$$

which reduces to the expression in 1e) after inserting the expressions for $E(X_k|x_0)$ and $\text{var}(X_k|x_0)$.

[1]There is an error in the expession in the exam from 2011.