Useful formulae

You will need the rules of double expectation and double variance; i.e.

$$E(Y) = E\{\xi(\mathbf{X})\} \qquad \text{og} \qquad \text{var}(Y) = E\{\sigma^2(\mathbf{X})\} + \text{var}\{\xi(\mathbf{X})\}$$

where $\xi(\mathbf{x}) = E(Y|\mathbf{X} = \mathbf{x})$ and $\sigma^2(\mathbf{x}) = \text{var}(Y|\mathbf{X} = \mathbf{x})$. There are also expressions for log-normal expectation and standard deviation you may draw on, i.e. if θ og τ are parametres and $\varepsilon \sim N(0,1)$, then

$$E(e^{\theta+\tau\varepsilon}) = e^{\theta+\tau^2/2} \qquad \text{og} \qquad \text{sd}(e^{\theta+\tau\varepsilon}) = e^{\theta+\tau^2/2}\sqrt{e^{\tau^2} - 1}.$$

Problem 1

Let $X_1, \ldots, X_J$ be payouts to J policies during one year and $\mathcal{X} = X_1 + \ldots + X_J$ the total for the entire portfolio. Suppose $X_1, \ldots, X_J$ have the same probability distribution.

a) Offer a simple matematicak argument which shows that for independent policies $\text{sd}(\mathcal{X})/E(\mathcal{X}) \to 0$ when $J \to \infty$.

b) Derive an expression for $\text{sd}(\mathcal{X})/E(\mathcal{X})$ when the risks $X_1, \ldots, X_J$ depend on a common, random factor ω and argue that the result in a) no longer holds. [**Hint:** Suppose $X_1, \ldots, X_J$ are independent given ω and use the rules of double expectation and double variance].

Let $\mathcal{N}$ be the number of claims against the portfolio so that

$$\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}$$

where $Z_1, Z_2, \ldots$ are losses per event. Assume that their model is the logistic one with distribution function

$$F(z) = 1 - \frac{1+\alpha}{1+\alpha e^{z/\beta}}, \qquad z > 0$$

where α og β are positive parametres.

c) You won't find a sampling procedure for this model in standard software. Design one yourself using inversion.

d) Write a simulation program for $\mathcal{X}$ when $\mathcal{N}$ is Poisson distributed with parameter $\lambda = J\mu T$.

Persentiles for $\mathcal{X}$ when $\lambda = 20$, $\alpha = 2$ og $\beta = 1$ is recorded in the table ($m = 100000$ simulations used).

$E(\mathcal{X})$	1%	5%	25%	50%	75%	90%	95%	99%
24.3	9.65	13.20	19.11	23.84	29.00	34.10	37.35	43.65

e) What's the reserve at 95% and 99%?

Suppose for each incident for which Z exceeds a an amount $Z - a$ is reimbursed through a reinsurance arrangement, though never more than a certain maximum b.

f) Modify the program in d) so that the re-insurer risk $\mathcal{X}^{\text{re}}$ is simulated.

Simulations for the same parametres as above gave when $a = 2$ and $b = 8$

$E(\mathcal{X})$	1%	5%	25%	50%	75%	90%	95%	99%
3.93	0	0.31	1.79	3.41	5.52	7.78	9.33	12.57

g) What is the pure premium of the re-insurance and what is the 95% og 99% reserve of the re-insurer?

Problem 2

The pensions below start at age l_r years and is paid as an amount s at the start of each year until the individual dies. Valuation of the scheme draws on tables ${}_kp_l$ of the probability of living k years longer at age l. A rate of interest r for discounting is needed too.

a) Write down an expression for the present value π_{l_0} of such a pension for an indvidual at age l_0 when $l_0 < l_r$.

b) The same question when $l_0 \geq l_r$.

The figure below plots π_{l_0} against the retirement age l_r between 55 og 70 years when $l_0 = 35$, $s = 1$ and the life table is

$${}_kp_l = \exp\left(-\theta_0 k - \frac{\theta_1}{\theta_2}(e^{\theta_2 k} - 1)e^{\theta_2 l}\right)$$

where $\theta_0 = 0.009$, $\theta_1 = 0.000046$ og $\theta_2 = 0.0908$. The rate of interest is varied between 0.02, 0.03, 0.04 og 0.05.

c) Use a sentence or two explain the pattern and identify which curve belongs to which rate of interest.

d) If the pension is financed by fixed contributions ζ at the start of each year from age l_0 up to $l_r - 1$ what is the present value of all these payments?

e) What does it mean that ζ is determined by equivalence and write down a mathematical expression for it.

Suppose there are N_l individuals in age l.

f) What is the total liability for all of them when future contributions are not counted?

Oppgave 3

a) If $R_1, \dots, R_K$ are the returns in K periods from a financial investment, what is the aggregated return $R_{0:K}$ for all the periods together?

b) What is the standard model for $R_1, \dots, R_K$ when investments are in the stock market?

c) Determine the probability distributoon for $R_{0:K}$ when the model in b) is log-normal.

The distribution of $R_{0:K}$ is much more complicated for assets other than equity. Suppose $R_k = r_k$ is floating rate of interest with $r_1, \dots, r_K$ following a log-normal, auto-regressive model of the form

$$r_k = \xi e^{-\frac{1}{2}\sigma^2/(1-a^2)+X_k} \qquad \text{where} \qquad X_k = aX_{k-1} + \sigma\varepsilon_k, \quad k = 1, \dots, K.$$

Here ξ, a og σ are parametres with $|a| < 1$ and $\varepsilon_1 \dots, \varepsilon_K$ independent and $N(0,1)$. The recursion starts at $X_0 = x_0$ where $x_0 = \log(r_0/\xi) + \frac{1}{2}\sigma^2/(1-a^2)$.

d) Write a program simulating $r_{0:K} = R_{0:K}$ under this model.

Take for granted that

$$E(X_k|x_0) = a^k x_0 \qquad \text{and} \qquad \text{sd}(X_k|x_0) = \sqrt{\frac{1-a^{2k}}{1-a^2}}\,\sigma.$$

e) Use this to verify that

$$E(r_k|r_0) = \xi e^{a^k x_0 - \frac{1}{2}\sigma^2 a^{2k}/(1-a^2)} \qquad \text{and} \qquad \text{sd}(r_k|r_0) = E(r_k|r_0)\sqrt{e^{\sigma^2(1-a^{2k})/(1-a^2)} - 1}.$$

f) Argue that when $k \to \infty$, then

$$E(r_k|r_0) \to \xi \qquad \text{og} \qquad \text{sd}(r_k|r_0) \to \xi\sqrt{e^{\sigma^2/(1-a^2)} - 1}.$$