

Obligatory assignment for STK3505/4505, Autumn 2016 – Solution

Part I

A non-life insurance company has responsibility for a portfolio of $J = 1,000$ policies. Further, the number of claims \mathcal{N} is Poisson distributed with intensity $\mu = 0.01$, and that the claim sizes Z_i are Gamma distributed with $E(Z_i) = 1.7$ and $\text{sd}(Z_i) = 1, 2, 3.5$.

a) The parameter ξ is given by:

$$\xi = E(Z_i) = \underline{1.7}.$$

The standard deviation of the Gamma distribution is given by:

$$\text{sd}(Z_i) = \frac{\xi}{\sqrt{\alpha}},$$

which results in:

$$\alpha = \frac{\xi^2}{(\text{sd}(Z_i))^2} = \begin{cases} \frac{1.7^2}{1^2} = \underline{2.89}, & \text{sd}(Z_i) = 1 \\ \frac{1.7^2}{2^2} \approx \underline{0.723}, & \text{sd}(Z_i) = 2 \\ \frac{1.7^2}{3.5^2} \approx \underline{0.236}, & \text{sd}(Z_i) = 3.5 \end{cases}.$$

The corresponding pdfs are shown in Figure 1. We see that the distribution becomes more heavy-tailed as α decreases.

b) We know that the total portfolio loss \mathcal{X} over a period of 1 year is given by

$$\mathcal{X} = Z_1 + \dots + Z_{\mathcal{N}},$$

where \mathcal{N} is the total number of claims and Z_i are the individual claim sizes. Further, $\mathcal{N} \sim \text{Poisson}(J\mu T)$, with $T = 1$ year, and $Z_i \sim \text{Gamma}(\alpha, \xi)$. Finally, we assume that the claims sizes Z_i are independent of each other and of \mathcal{N} . An estimate of the $\epsilon \cdot 100\%$ reserve q_ϵ for this portfolio given by

$$P(\mathcal{X} \leq q_\epsilon) = \epsilon,$$

is then obtained by generating m samples $\mathcal{X}_1^*, \dots, \mathcal{X}_m^*$, ordering them in ascending order as $\mathcal{X}_{(1)}^* \leq \dots \leq \mathcal{X}_{(m)}^*$, and letting $\hat{q}_\epsilon = \mathcal{X}_{(\epsilon m)}^*$. To ensure

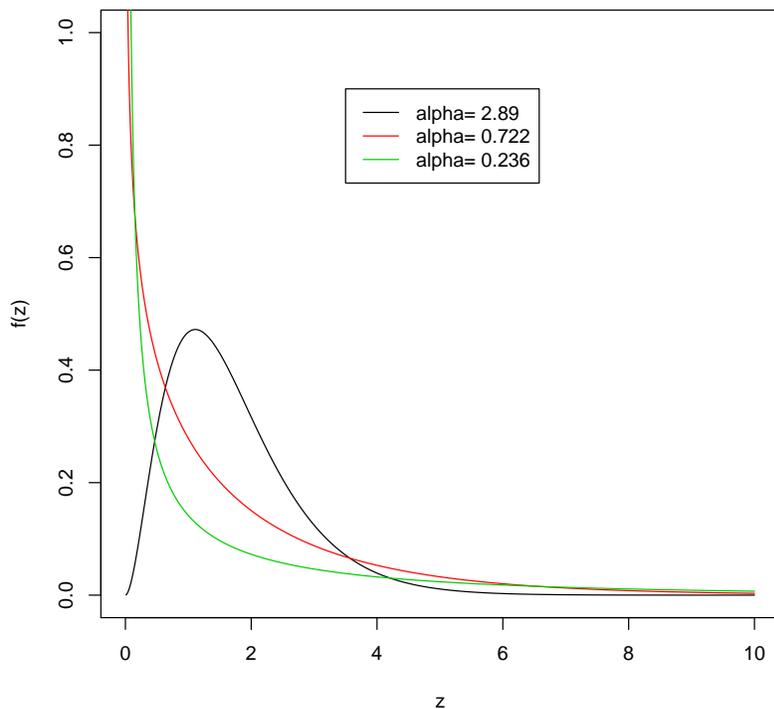


Figure 1: Probability density function of each of the three distributions.

that the Monte Carlo error is small, we use a large m , more specifically $m = 100,000$. The resulting estimates of the reserve are given in Table . We see that the reserve increases as α decreases, which is due to that fact that the claim size distribution, and consequently the distribution of \mathcal{X} , becomes more heavy-tailed. The relative difference between the reserves from the three distributions are larger at level 99% than at level 95% for the same reason.

c) When the actual compensations have a deductible a and a maximum insured sum b per claim, the total portfolio loss becomes

$$\mathcal{X} = H(Z_1) + \dots + H(Z_N),$$

Reserve	$\alpha = 2.89$	$\alpha = 0.723$	$\alpha = 0.236$
95%	28.0	32.3	40.9
99%	33.5	41.0	56.4

Table 1: 95% and 99% reserve for the portfolio for each of the three sets of parameters.

Reserve	$\alpha = 2.89$	$\alpha = 0.723$	$\alpha = 0.236$
95%	19.2	20.2	17.7
99%	23.4	25.1	22.7

Table 2: 95% and 99% reserve for the portfolio for each of the three sets of parameters when the actual compensations have a deductible a and a maximum insured sum b per claim.

where

$$H(z) = \begin{cases} 0, & z < a \\ z - a, & a \leq z \leq a + b \\ b, & z > a + b \end{cases}$$

Taking $H(\cdot)$ into account in the simulation of \mathcal{X} , the reserve can be estimated as described in b). The resulting values are given in Table . We see that the reserves are smaller than in b). That is due to the fact that all claims are reduced by the deductible and that the smallest claims are truncated by the deductible and the largest claims by the maximum insured sum b . These effects are largest for the most distribution with the heaviest tails, i.e. with $\alpha = 0.236$, since it is the distribution with the largest number of claims above $a + b$ and below a .

Part II

The net assets of the company are given by the recursion

$$\mathcal{Y}_k = (1 + \mathcal{R}_k)\mathcal{Y}_{k-1} + \Pi_k - \mathcal{O}_k - \mathcal{X}_k, \quad k = 1, 2, \dots$$

with $\mathcal{Y}_0 = \nu_0$, $\text{sd}(Z_i) = 3.5$ and $\Pi_k - \mathcal{O}_k = 18$, for all k . Now, define

$$\underline{\mathcal{Y}} = \min(\mathcal{Y}_1, \dots, \mathcal{Y}_K).$$

The ruin capital at the horizon K years with ruin probability ϵ is then the start capital $\nu_{0\epsilon}$ that satisfies

$$p^{ru}(\nu_{0\epsilon}) = \mathbb{P}(\underline{\mathcal{Y}} < 0 | \mathcal{Y}_0 = \nu_{0\epsilon}).$$

a) We start by assuming that there is no financial income ($\mathcal{R}_k = 0$, for all k). Then,

$$\mathcal{Y}_k = \mathcal{Y}_{k-1} + \Pi_k - \mathcal{O}_k - \mathcal{X}_k,$$

and a Monte Carlo estimate of $\nu_{0\epsilon}$ is found by

1. generating m samples $\underline{\mathcal{Y}}_1^*, \dots, \underline{\mathcal{Y}}_m^*$ of $\underline{\mathcal{Y}}$ from the above model with start capital $\nu_0 = 0$
2. ordering the samples in ascending order $\underline{\mathcal{Y}}_{(1)}^* \leq \dots \leq \underline{\mathcal{Y}}_{(m)}^*$
3. letting $\nu_{0\epsilon}^* = -\underline{\mathcal{Y}}_{(\epsilon m)}^*$.

With $m = 100,000$, we obtain the results in the first column of Table . We see that the ruin capital increases substantially when the ruin probability is set to 1% instead of 5%. This is due to the fact that the distribution of the claim sizes Z and thus the aggregated loss \mathcal{X} is quite heavy-tailed.

b) Repeating the above procedure to estimate the ruin capital at the shorter horizons $K = 10$ and $K = 5$, we obtain the results in columns 2 and 3 of Table , respectively. When the time horizon is reduced, so is the ruin capital, since there are fewer years during which the net assets may fall below 0. As for $K = 20$, the ruin capital increases significantly when the ruin probability is set to 1% instead of 5%.

c) Now, we will assume that there is financial income with return $\mathcal{R}_k = e^{\xi_R + \sigma_R \varepsilon} - 1$, where $\varepsilon \sim N(0, 1)$, $\xi_R = 0.07$ and $\sigma_R = 0.2$. It is easily shown that the nets assets at time k may be written as

$$\mathcal{Y}_k = (1 + \mathcal{R}_{0:k})(\nu_0 - \mathcal{S}_k),$$

where

$$\mathcal{S}_k = \sum_{i=1}^k \frac{\mathcal{X}_i + \mathcal{O}_i - \Pi_i}{1 + \mathcal{R}_{0:i}}.$$

ϵ	$K = 20$	$K = 10$	$K = 5$
5%	92.7	68.4	49.9
1%	134	99.8	74.3

Table 3: Start capital $\nu_{0\epsilon}$ for ruin probabilities $\epsilon = 5\%$, 1% at horizons $K = 20$, 10 and 5 years when there is no financial income.

ϵ	$K = 20$	$K = 10$	$K = 5$
5%	69.8	58.8	45.4
1%	118	95.6	72.7

Table 4: Start capital $\nu_{0\epsilon}$ for ruin probabilities $\epsilon = 5\%$, 1% at horizons $K = 20$, 10 and 5 years when there is financial income.

The procedure for estimating $\nu_{0\epsilon}$ is now different. First define

$$\bar{\mathcal{S}} = \max(\mathcal{S}_1, \dots, \mathcal{S}_K).$$

Then,

1. generate m samples $\bar{\mathcal{S}}_1^*, \dots, \bar{\mathcal{S}}_m^*$ of $\bar{\mathcal{S}}$ from the above model
2. order the samples in ascending order $\bar{\mathcal{S}}_{(1)}^* \leq \dots \leq \bar{\mathcal{S}}_{(m)}^*$
3. let $\nu_{0\epsilon}^* = \bar{\mathcal{S}}_{((1-\epsilon)m)}^*$.

With $m = 100,000$, we obtain the results in the first column of Table . We see that the ruin capital is substantially reduced when there is financial income. The effect becomes stronger for longer horizons K , as the financial income has more time to accumulate.

R-code

```
## Part I
```

```
sigma1 <- 1
sigma2 <- 2
sigma3 <- 3.5
mu <- 0.01
```

```

T <- 1
J <- 1000

# a)

xi <- 1.7
alpha1 <- xi^2/sigma1^2
alpha2 <- xi^2/sigma2^2
alpha3 <- xi^2/sigma3^2

z <- seq(0.01,10,0.01)
plot(z,dgamma(z/xi,alpha1,alpha1)/xi,type="l",ylim=c(0,1),xlab="z",
ylab="f(z)")
lines(z,dgamma(z/xi,alpha2,alpha2)/xi,col=2)
lines(z,dgamma(z/xi,alpha3,alpha3)/xi,col=3)
legend(3.5,0.9,paste("alpha=",round(c(alpha1,alpha2,alpha3),3)),
lty=rep(1,3),col=1:3)

# Alternative:

m <- 1e6
Z1 <- xi*rgamma(m,alpha1,alpha1)
Z2 <- xi*rgamma(m,alpha2,alpha2)
Z3 <- xi*rgamma(m,alpha3,alpha3)

plot(density(Z1,from=0.01),ylim=c(0,1),xlab="z",ylab="f(z)")
lines(density(Z2,from=0.01),col=2)
lines(density(Z3,from=0.01),col=3)
legend(3.5,0.9,paste("alpha=",round(c(alpha1,alpha2,alpha3),3)),
lty=rep(1,3),col=1:3)

# b)

m <- 1e5
yps <- c(0.95,0.99)
Ncal <- rpois(m,J*mu*T)
Xcal1 <- Xcal2 <- Xcal3 <- rep(0,m)
for(i in 1:m)

```

```

{
  Z1 <- xi*rgamma(Ncal[i],alpha1,alpha1)
  Z2 <- xi*rgamma(Ncal[i],alpha2,alpha2)
  Z3 <- xi*rgamma(Ncal[i],alpha3,alpha3)
  Xcal1[i] <- sum(Z1)
  Xcal2[i] <- sum(Z2)
  Xcal3[i] <- sum(Z3)
}
sort(Xcal1)[m*yps]
sort(Xcal2)[m*yps]
sort(Xcal3)[m*yps]

# c)

a <- 0.6
b <- 4
Xcal1 <- Xcal2 <- Xcal3 <- rep(0,m)
for(i in 1:m)
{
  Z1 <- xi*rgamma(Ncal[i],alpha1,alpha1)
  Z2 <- xi*rgamma(Ncal[i],alpha2,alpha2)
  Z3 <- xi*rgamma(Ncal[i],alpha3,alpha3)
  Xcal1[i] <- sum(pmin(pmax(Z1-a,0),b))
  Xcal2[i] <- sum(pmin(pmax(Z2-a,0),b))
  Xcal3[i] <- sum(pmin(pmax(Z3-a,0),b))
}
sort(Xcal1)[m*yps]
sort(Xcal2)[m*yps]
sort(Xcal3)[m*yps]

## Part II

# a)

Pi_0 <- 18
nu0 <- 0
K <- 20
yps <- c(0.05,0.01)

```

```

m <- 1e5

N <- rpois(m*K, J*mu*T)
Xcal <- rep(0, K*m)
for(i in 1:(m*K))
  {
    Z <- xi*rgamma(N[i], alpha3, alpha3)
    Xcal[i] <- sum(Z)
  }
Xcal <- matrix(Xcal, K, m)
Ycal <- nu0+(1:K)*Pi_0-apply(Xcal, 2, cumsum)
Ycal <- rbind(rep(nu0, m), Ycal)
Ymin <- apply(Ycal, 2, min)
nu0yps <- -sort(Ymin)[m*yyps]
nu0yps

# b)

Ymin <- apply(Ycal[0:10+1,], 2, min)
nu0yps <- -sort(Ymin)[m*yyps]
nu0yps

Ymin <- apply(Ycal[0:5+1,], 2, min)
nu0yps <- -sort(Ymin)[m*yyps]
nu0yps

# c)

xiR <- 0.07
sigmaR <- 0.2
Rcal <- rep(0, m)
Scal <- matrix(0, K+1, m)
for(k in 1:K)
  {
    eps <- rnorm(m)
    Rcal <- (1+Rcal)*exp(xiR+sigmaR*eps)-1
    Scal[k+1,] <- Scal[k,]+(Xcal[k,]-Pi_0)/(1+Rcal)
  }

```

```
Scalmax <- apply(Scal,2,max)
nu0yps <- sort(Scalmax)[(1-yps)*m]
nu0yps
```

```
Scalmax <- apply(Scal[0:10+1,],2,max)
nu0yps <- sort(Scalmax)[(1-yps)*m]
nu0yps
```

```
Scalmax <- apply(Scal[0:5+1,],2,max)
nu0yps <- sort(Scalmax)[(1-yps)*m]
nu0yps
```