STK 3505/4505: Summary of the course

November 22, 2016

CH 2: Getting started the Monte Carlo Way

- How to use Monte Carlo methods for estimating quantities ψ related to the distribution of X, based on the simulations $X_1^*, \ldots, X_m^*$:
  - mean: $\bar{X}^* = \frac{1}{m}\sum_{i=1}^m X_i^*$
  - standard deviation: $s^* = \sqrt{\frac{1}{m-1}\sum_{i=1}^m \left(X_i^* - \bar{X}^*\right)^2}$
  - ϵ -percentile q_ϵ given by $\mathrm{P}(X \leq q_\epsilon) = \epsilon$: $q_\epsilon^* = X_{(m\epsilon)}^*$, where $X_{(1)}^* \leq \ldots \leq X_{(m)}^*$.
- Inversion method for sampling X with the cdf $F(x)$:

  0. Input: parameters of $F(x)$
  1. Draw $U^* \sim U(0,1)$
  2. Return $X^* \leftarrow F^{-1}(U^*)$ % or $F^{-1}(1-U^*)$

CH 2: Getting started the Monte Carlo Way

- The normal distribution and extensions:
  - transformations, such as the log-normal $Y = e^X$, $X \sim N(\xi, \sigma)$.
  - stochastic volatility: $X = \xi + \xi_\sigma \sqrt{Z}\epsilon$, with $\epsilon \sim N(0,1)$ and $Z > 0$ independent of ϵ, for instance $Z = \frac{1}{G}$, $G \sim Gamma(\alpha)$.
  - multivariate: bivariate, equicorrelation model.
- Positive distributions used in insurance:
  - continuous: log-normal, Gamma, Exponential, Weibull, Pareto
  - discrete: Poisson.

CH 3: Evaluating risk: A primer

- Models for general insurance:
  - claim numbers N (policy) and $\mathcal{N}$ (portfolio of J policies) during a period of length T, typically $N \sim Poisson(\mu T)$ and $\mathcal{N} \sim Poisson(J\mu T)$.
  - claim sizes $Z_i > 0$, typically iid and independent of $N/\mathcal{N}$.
  - aggregate compensations: $X = Z_1 + \ldots + Z_N$ (policy) and $\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}$ (portfolio).
  - compensation functions: $X = H(Z_1) + \ldots + H(Z_N)$, $\mathcal{X} = H(Z_1) + \ldots + H(Z_{\mathcal{N}})$, with $0 \leq H(z) \leq z$, for instance

$$H(z) = \begin{cases} 0, & z < a \\ z - a, & a \leq z < a + b \\ b, & z \geq a + b \end{cases} .$$

  - reinsurance: $Z_i^{re} = H(Z_i)$ (contract at policy level) or $\mathcal{X}^{re} = H(\mathcal{X})$ (contract at portfolio level), where $0 \leq H(z) \leq z$, for instance as above.

CH 3: Evaluating risk: A primer

- Models for life insurance:
  - sequence of equidistant payments $X_1, X_2, \ldots$
  - k-step survival probability ${}_k p_l$: probability of surviving till age at least $k + l$ given current age l ${}_k p_l = p_{l+k-1} \cdot p_{l+k-2} \cdot \ldots \cdot p_l$, where $p_j =_1 p_j$.
  - expected present value of the payments: $V_0 = \sum_{k=0}^{\infty} d^k \mathrm{E}(X_k)$, typically with $d = 1/(1+r)$.
  - pension scheme with start age l_0, yearly premium π before retirement age l_r, yearly payment s from age l_r: $V_0 = -\pi \sum_{k=0}^{l_r - l_0 - 1} d_k^k p_{l_0} + s \sum_{k=l_r - l_0}^{\infty} d_k^k p_{l_0}$.
  - equivalence premium: π given by $V_0 = 0$.
  - One-time premium: π_{l_0} given by $V_0 = 0$ when the premium is paid only once, at age l_0.

CH 3: Evaluating risk: A primer

- Protecting financial investments with options:
  - pay-off $X = H(\mathcal{R})\nu_0$ at expiry T, when ν_0 is the investment and $\mathcal{R}$ is the return of the asset during T.
  - put-option: $H(\mathcal{R}) = \max(r_g - \mathcal{R}, 0)$, call option: $H(\mathcal{R}) = \max(\mathcal{R} - r_g, 0)$, both with guarantee r_g.
  - risk-neutral price π: $\pi = e^{-rT} E_Q(X)$, where e^{-rT} is the discount over T and Q is the modified, risk-neutral model
    - Black-Scholes: exact formula for π when $\mathcal{R} = e^{\xi T + \sigma\sqrt{T}\epsilon} - 1$, $\epsilon \sim N(0,1)$. Under Q $\mathcal{R} = e^{rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\epsilon} - 1$.

$$\pi(\nu_0) = (e^{-rT}(1 + r_g)\Phi(a) - \Phi(a - \sigma\sqrt{T}))\nu_0$$

$$a = \frac{\log(1 + r_g) - rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

    - by simulation: also for other models for $\mathcal{R}$.

1. Simulate $X_1^*, \ldots, X_m^*$ under Q

$$2. \pi^* = \frac{e^{-rT}}{m} \sum_{i=1}^{m} X_i^*.$$

- Net assets in general insurance and the ruin problem:
    - recursion for net assets $\mathcal{Y}_k$:

$$\mathcal{Y}_k = (1 + \mathcal{R}_k)\mathcal{Y}_{k-1} + \Pi_k - \mathcal{O}_k - \mathcal{X}_k,$$

      where $\mathcal{Y}_0 = \nu_0$, $\mathcal{R}_k$ is the return from the financial investments, Π_k the total premium, $\mathcal{O}_k$ the overhead and $\mathcal{X}_k$ the total pay-off to the portfolio of policies.
    - ruin problem: finding the start capital $\nu_{0\epsilon}$, such that $\mathrm{P}(\min(\mathcal{Y}_1, \ldots, \mathcal{Y}_K) < 0 | \mathcal{Y}_0 = \nu_{0\epsilon}) = \epsilon$.

CH 5: Modelling I: Linear dependence

- Markowitz optimal portfolio selection:
  - $R_1, \ldots, R_J$ returns of J assets with means $\xi_1, \ldots, \xi_J$ and covariance matrix $\boldsymbol{\Sigma}$
  - r risk-free opportunity
  - $\mathcal{R} = w_0 r + w_1 R_1 + \ldots + R_J$ portfolio return with $w_0 + w_1 + \ldots + w_J = 1$
  - objective: find $w_0, w_1, \ldots, w_J$ so that $\text{Var}(\mathcal{R}) = \sum_{i=1}^J \sum_{j=1}^J w_i w_j \sigma_{ij}$ is minimised when $\text{E}(\mathcal{R}) = r + \sum_{j=1}^J w_j(\xi_j - r) = e_g$
  - solution: $w_j = \gamma \tilde{w}_j$, $j = 1, \ldots, J$, $w_0 = 1 - \sum_{j=1}^J w_j$, with $\sum_{j=1}^J \tilde{w}_j \sigma_{ij} = \xi_i - r$, $i = 1, \ldots, J$ and $\gamma = \frac{e_g - r}{\sum_{j=1}^J \tilde{w}_j(\xi_i - r)}$.

CH 5: Modelling I: Linear dependence

- Multivariate normal distribution with extensions:
  - simulation:

    0. Input: $\boldsymbol{\xi}, \boldsymbol{C}$
    1. Draw $\eta_1^*, \ldots, \eta_J^* \overset{iid}{\sim} N(0,1)$
    2. Return $\boldsymbol{X}^* \leftarrow \boldsymbol{\xi} + \boldsymbol{C}\boldsymbol{\eta}^*$,

    where $\boldsymbol{\eta}^* = (\eta_1^*, \ldots, \eta_J^*)^T$ and $\boldsymbol{\Sigma} = \boldsymbol{C}\boldsymbol{C}^T$.
  - transformations: multivariate log-normal model
  - stochastic volatility.

CH 5: Modelling I: Linear dependence

- Random walk:
  - standard: $Y_k = Y_{k-1} + X_k$, $X_k = \xi + \sigma\epsilon_k$, $\epsilon_k \overset{iid}{\sim} N(0,1)$, $k = 1, 2, \ldots$
  - geometric: $S_k = e^{Y_k} = (1 + R_k)S_{k-1}$, $\log(1 + R_k) = X_k$, $k = 1, 2, \ldots$
  - after k periods: $Y_k = Y_0 + k\xi + \sqrt{k}\sigma\eta_k$, $S_k = S_0 e^{k\xi + \sqrt{k}\sigma\eta_k}$, $\eta_k \sim N(0,1)$
  - multivariate
  - simulation.
- Mean-reversion models:
  - AR(1)-model: $Y_k = \xi + X_k$, $X_k = aX_{k-1} + \sigma\epsilon_k$, $\epsilon_k \overset{iid}{\sim} N(0,1)$, $X_0 = x_0$, $k = 1, 2, \ldots$, with $|a| < 1$ for stationarity
  - after k periods: $\mathrm{E}(X_k|x_0) = a^k x_0 \underset{k\to\infty}{\longrightarrow} 0$

    $\mathrm{sd}(X_k|x_0) = \sqrt{\frac{1-a^{2k}}{1-a^2}}\sigma \underset{k\to\infty}{\longrightarrow} \frac{\sigma}{\sqrt{1-a^2}}$

    $\mathrm{Cov}(X_k, X_{k-l}|x_0) = a^l \frac{1-a^{2k}}{1-a^2}\sigma^2 \underset{k\to\infty}{\longrightarrow} a^l \frac{\sigma^2}{1-a^2}$
  - transformations
  - simulation.

CH 6: Modelling II: Conditional and non-linear

- Conditional models:
  - preservation of normality under conditioning
  - survival probabilities: ${}_k p_l = \mathrm{P}(Y > l + k | Y > l)$, where Y is the age at which a person dies
  - extreme claim sizes
- Rules of double expectation and variance:
  - $$\mathrm{E}(Y) = \mathrm{E}(\mathrm{E}(Y|\boldsymbol{X}))$$
    $$\mathrm{Var}(Y) = \mathrm{Var}(\mathrm{E}(Y|\boldsymbol{X})) + \mathrm{E}(\mathrm{Var}(Y|\boldsymbol{X}))$$
  - Example: $\mathcal{N} \sim \mathit{Poisson}(J\mu T)$, $Z_1, \ldots, Z_{\mathcal{N}}$ iid with $\mathrm{E}(Z_i) = \xi_z$, $\mathrm{sd}(Z_i) = \sigma_z$

$$\mathrm{E}(\mathcal{X}|\mathcal{N}) = \mathcal{N}\xi_z, \quad \mathrm{Var}(\mathcal{X}|\mathcal{N}) = \mathcal{N}\sigma_z^2$$
$$\mathrm{E}(\mathcal{X}) = \mathrm{E}(\mathrm{E}(\mathcal{X}|\mathcal{N})) = J\mu T\xi_z$$
$$\mathrm{Var}(\mathcal{X}) = \mathrm{Var}(\mathrm{E}(\mathcal{X}|\mathcal{N})) + \mathrm{E}(\mathrm{Var}(\mathcal{X}|\mathcal{N})) = J\mu T(\sigma_z^2 + \xi_z^2).$$

CH 6: Modelling II: Conditional and non-linear

- Common factor models:
  - $X_1, \ldots, X_J$ all depend on a common, stochastic factor ω
  - $X_1, \ldots, X_J$ iid, conditionally on ω
  - Risk cannot be diversified away
  - Examples: claim counts with a common, stochastic intensity μ; CAPM
- Simulation of multivariate models based on conditioning:
  - Draw X_1^* from $f(x_1)$
  - Draw X_2^* from $f(x_2|X_1^*)$
  - Draw X_3^* from $f(x_3|X_1^*, X_2^*)$
  - $\vdots$
  - Draw X_n^* from $f(x_n|X_1^*, \ldots, X_{n-1}^*)$.