

## Obligatory assignment for STK3505/4505, Autumn 2017 – Solution

### Part I

A non-life insurance company has responsibility for a portfolio of  $J = 1,000$  policies. Further, the number of claims  $\mathcal{N}$  is Poisson distributed with intensity  $\mu = 0.01$ , and that the claim sizes  $Z_i$  are log-normal distributed with  $E(Z_i) = 1.9$  and  $sd(Z_i) = 1, 2, 3.5$ , which can be represented as  $\log(Z_i) = \xi + \sigma\epsilon$ , where  $\epsilon \sim N(0, 1)$

a) The mean and standard deviation of the log-normal distribution are given by:

$$E(Z_i) = \exp(\xi + \sigma^2/2) = \underline{1.9},$$
$$sd(Z_i) = E(Z_i)\sqrt{(e^{\sigma^2} - 1)}.$$

which results in:

$$\begin{cases} \sigma_1 = \sqrt{\ln((1/1.9)^2 + 1)} \approx \underline{0.49}, & sd(Z_i) = 1 \\ \sigma_2 = \sqrt{\ln((2/1.9)^2 + 1)} \approx \underline{0.86}, & sd(Z_i) = 2 \\ \sigma_3 = \sqrt{\ln((3.5/1.9)^2 + 1)} \approx \underline{1.22}, & sd(Z_i) = 3.5 \end{cases} .$$

$$\begin{cases} \xi_1 = \ln(1.9) - \sigma_1^2/2 \approx \underline{0.52}, & \sigma_1 \approx 0.49 \\ \xi_2 = \ln(1.9) - \sigma_2^2/2 \approx \underline{0.27}, & \sigma_2 \approx 0.86 \\ \xi_3 = \ln(1.9) - \sigma_3^2/2 \approx \underline{-0.10}, & \sigma_3 \approx 1.22 \end{cases} .$$

The corresponding pdfs are shown in Figure 1. We see that the distribution becomes more heavy-tailed as  $\sigma$  increases.

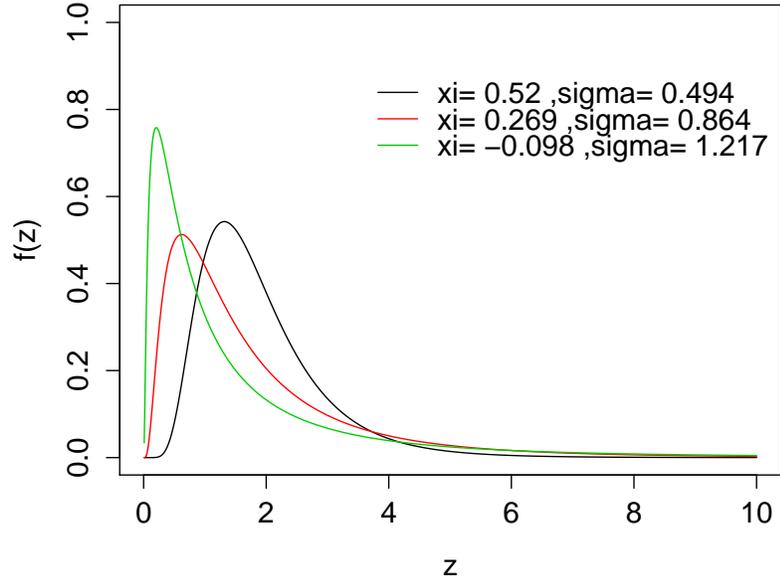


Figure 1: Probability density function of each of the three distributions.

b) We know that the total portfolio loss  $\mathcal{X}$  over a period of 1 year is given by

$$\mathcal{X} = Z_1 + \dots + Z_{\mathcal{N}},$$

where  $\mathcal{N}$  is the total number of claims and  $Z_i$  are the individual claim sizes. Further,  $\mathcal{N} \sim \text{Poisson}(J\mu T)$ , with  $T = 1$  year, and  $Z_i$  is lognormal distributed. Finally, we assume that the claims sizes  $Z_i$  are independent of each other and of  $\mathcal{N}$ . An estimate of the  $\epsilon \cdot 100\%$  reserve  $q_\epsilon$  for this portfolio given by

$$P(\mathcal{X} \leq q_\epsilon) = \epsilon,$$

is then obtained by generating  $m$  samples  $\mathcal{X}_1^*, \dots, \mathcal{X}_m^*$ , ordering them in ascending order as  $\mathcal{X}_{(1)}^* \leq \dots \leq \mathcal{X}_{(m)}^*$ , and letting  $\hat{q}_\epsilon = \mathcal{X}_{(em)}^*$ . The simulation algorithm for  $\mathcal{X}$  is

- 1: Input:  $J, \mu, T, \xi, \sigma, m$
- 2: **for**  $i=1, \dots, m$  **do**
- 3:     Draw  $\mathcal{N}^* \sim \text{Poisson}(J\mu T)$

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4:    $\mathcal{X}_i^* \leftarrow 0$ 
5:   for  $j=1, \dots, \mathcal{N}^*$  do
6:     Draw  $\epsilon^* \sim N(0, 1)$ 
7:      $Z^* \leftarrow e^{\xi + \sigma \epsilon^*}$ 
8:      $\mathcal{X}_i^* \leftarrow \mathcal{X}_i^* + Z^*$ 
9:   end for
10: end for
11: Return  $\mathcal{X}_1^*, \dots, \mathcal{X}_m^*$ .

```

To ensure that the Monte Carlo error is small, we use a large  $m$ , more specifically  $m = 1,000,000$ . The resulting estimates of the reserve are given in Table 1. We see that the reserve increases as  $\sigma$  increases, which is due to that fact that the claim size distribution, and consequently the distribution of  $\mathcal{X}$ , becomes more heavy-tailed. The relative difference between the reserves from the three distributions are larger at level 99% than at level 95% for the same reason.

Reserve	$\xi = 0.52, \sigma = 0.49$	$\xi = 0.27, \sigma = 0.86$	$\xi = -0.10, \sigma = 1.22$
95%	30.99	35.02	41.49
99%	36.96	44.90	62.26

Table 1: 95% and 99% reserve for the portfolio for each of the three sets of parameters.

c) Now we spread the risk by applying a layer  $a \times b$  reinsurance contract to the individual losses  $Z_i$  such that the cedent keeps all risk below the retention limit  $a$  and covers the excess amount above the threshold  $b$ . The reinsurer and cedent responsibilities then become

$$\mathcal{X}^{re} = H(Z_1) + \dots + H(Z_{\mathcal{N}}),$$

$$\mathcal{X}^{ce} = \mathcal{X} - \mathcal{X}^{re},$$

where

$$H(z) = \begin{cases} 0, & z < a \\ z - a, & a \leq z \leq a + b \\ b, & z > a + b \end{cases}$$

Taking  $H(\cdot)$  into account in the simulation of  $\mathcal{X}$ , the reserve can be estimated as described in b). The results of reserve for cedent are given in Table 2. We see that the reserves are smaller than in b). That is due to the fact that some of the responsibilities are passed on to the reinsurer thus reducing the risk of the cedent. These effects are largest for the distribution with the lightest tails, i.e. with  $\sigma = 0.49$ , since it is the distribution with the largest number of claims above  $a$  and below  $a + b$ . In other words, the reinsurer covers more claims if they are from a less heavy-tailed distribution.

Reserve	$\xi = 0.52, \sigma = 0.49$	$\xi = 0.27, \sigma = 0.86$	$\xi = -0.10, \sigma = 1.22$
95%	11.22	16.87	26.28
99%	13.57	24.87	46.58

Table 2: 95% and 99% reserve for cedent for the portfolio for each of the three sets of parameters when the reinsurance is a layer  $a \times b$  contract.

There is no exact formula for the pure reinsurance premium  $\pi^{\text{re}} = E\{H(\mathcal{X})\}$ , but when we use Monte Carlo, it is approximated by

$$\pi^{\text{re}*} = \frac{1}{m} \sum_{i=1}^m H(\mathcal{X}_i^*).$$

The results are given in Table 3. We see that the reinsurance premium decreases as the claim distribution becomes more heavy-tailed. Since there are more claims bounded by the threshold  $b$  for heavier-tailed distributions, the responsibility of the reinsurer is reduced in this situation, and hence so is the pure reinsurance premium.

	$\xi = 0.52, \sigma = 0.49$	$\xi = 0.27, \sigma = 0.86$	$\xi = -0.10, \sigma = 1.22$
$\pi^{\text{re}}$	11.82	10.60	8.95

Table 3: The reinsurance premium for the portfolio for each of the three sets of parameters when the reinsurance is a layer  $a \times b$  contract.

## Part II

a) The payoff for a put option is given as

$$X = \max(r_g - R, 0)\nu_0,$$

where  $r_g$  is fixed by the contract and  $\nu_0$  is the initial value of the underlying asset. The rate of return  $R$  follows a lognormal distribution with volatility  $\sigma\sqrt{T}$ , which can be represented as

$$\log(1 + R) = \xi T + \sigma\sqrt{T}\epsilon, \quad \text{where } \epsilon \sim N(0, 1).$$

Under risk-neutral world, the put option premium is given by

$$\pi_P(\nu_0) = E_Q\{\max(r_g - R, 0)\}\nu_0, \quad \text{where } R = e^{\xi_q T + \sigma\sqrt{T}\epsilon} - 1,$$

We replace  $\xi$  by  $\xi_q = r - \frac{1}{2}\sigma^2$  under risk-neutral model. A closed formula for this put option premium is given by

$$\pi_P(\nu_0) = \{(1 + r_g)e^{-rT}\Phi(a) - \Phi(a - \sigma\sqrt{T})\}\nu_0$$

where

$$a = \frac{\log(1 + r_g) - rT + \sigma^2 T/2}{\sigma\sqrt{T}} \quad \text{and} \quad \Phi(x) \quad \text{is the standard normal integral.}$$

We can calculate the premium for put option both analytically and using Monte Carlo. A Monte Carlo way of doing this is found by

- 1: Input:  $\sigma, T, r_g, r, \nu_0$
- 2: Draw  $\epsilon^* \sim N(0, 1)$
- 3:  $R^* \leftarrow e^{rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\epsilon^*}$
- 4: Return  $X^* \leftarrow (r_g - R_i^*)\nu_0$

The option premium is approximately the discounted average

$$\pi_P^* = \frac{e^{-rT}}{m} \sum_{i=1}^m X_i^*.$$

Assuming  $\nu_0 = 1$  and with  $m=100,000$ , we obtain the results in the first row of Table 4. The analytic way to compute option premium is based on the Black-Scholes formula aforementioned and the results is shown in the second row of Table 4. We see that higher volatility increases the uncertainty and

makes the premium higher. Also, with  $m$  being large, the difference between analytically computation and using Monte Carlo method is minor.

	$\sigma = 0.2$	$\sigma = 0.25$	$\sigma = 0.3$
$\pi_P^*$	0.0907	0.1108	0.1308
$\pi_P$	0.0902	0.1102	0.1301

Table 4: the option premium for each of the three sets of parameters  $\sigma$  with unit initial value of underlying asset analytically and by using Monte Carlo method.

b) The cliquet options, have pay off

$$H(R) = \begin{cases} r_g - R, & R \leq r_g \\ 0, & r_g < R \leq r_c . \\ -(R - r_c), & R > r_c \end{cases}$$

By using the algorithm in **a)** with  $H(R)$ , we can generate  $m = 100,000$  samples of cliquet payoff and the approximated option premium. We see that the option premium is reduced when we have a cliquet option and it allows the option seller to keep the top of the return. With fixed ceiling  $r_c$ , the price does not increase too much as volatility increases. When the ceiling  $r_c = 0.15$  and the volatility 0.25, the premium of the cliquet is close to half that of the put option.

	$\sigma = 0.2$	$\sigma = 0.25$	$\sigma = 0.3$
$r_c = 0.06$	0.0236	0.0237	0.0238
$r_c = 0.1$	0.0379	0.0384	0.0385
$r_c = 0.15$	0.0522	0.0537	0.0545
$r_c = 0.3$	0.0772	0.0842	0.0888
$r_c = 0.4$	0.0845	0.0954	0.1030

Table 5

Table 6: the option premium for different  $\sigma$  and  $r_c$  with unit initial value of underlying asset by using Monte Carlo methods.

## R-code

```
## Part I
sd1 <- 1
sd2 <- 2
sd3 <- 3.5
mu <- 0.01
T <- 1
J <- 1000

# a)
meanall <- 1.9
sigma1 <- sqrt(log((sd1/meanall)^2+1))
sigma2 <- sqrt(log((sd2/meanall)^2+1))
sigma3 <- sqrt(log((sd3/meanall)^2+1))

xi1 <- log(meanall)- sigma1^2/2
xi2 <- log(meanall)- sigma2^2/2
xi3 <- log(meanall)- sigma3^2/2

z <- seq(0.01,10,0.01)
plot(z,dlnorm(z,xi1,sigma1),type="l",ylim=c(0,1),xlab="z",ylab="f(z)")
lines(z,dlnorm(z,xi2,sigma2),col=2)
lines(z,dlnorm(z,xi3,sigma3),col=3)
legend(3.5,0.9,paste("xi=",round(c(xi1,xi2,xi3),3),"",sigma=",",
round(c(sigma1,sigma2,sigma3),3)),lty=rep(1,3),col=1:3,bty = "n" )

# Alternative:

m <- 1e6
Z1 <- rlnorm(m,xi1,sigma1)
Z2 <- rlnorm(m,xi2,sigma2)
Z3 <- rlnorm(m,xi3,sigma3)

plot(density(Z1,from=0.01),xlab="z",ylab="f(z)",ylim=c(0,1))
lines(density(Z2,from=0.01),col=2)
lines(density(Z3,from=0.01),col=3)
```

```

legend(3.5,0.9,paste("xi=",round(c(xi1,xi2,xi3),3),"",sigma=",
round(c(sigma1,sigma2,sigma3),3)),lty=rep(1,3),col=1:3)

```

```

# b)

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```

yps <- c(0.95,0.99)
Ncal <- rpois(m,J*mu*T)
Xcal1 <- Xcal2 <- Xcal3 <- rep(0,m)
for(i in 1:m)
{
  Z1 <- rlnorm(Ncal[i],xi1,sigma1)
  Z2 <- rlnorm(Ncal[i],xi2,sigma2)
  Z3 <- rlnorm(Ncal[i],xi3,sigma3)
  Xcal1[i] <- sum(Z1)
  Xcal2[i] <- sum(Z2)
  Xcal3[i] <- sum(Z3)
}
sort(Xcal1)[m*yps]
sort(Xcal2)[m*yps]
sort(Xcal3)[m*yps]

```

```

# c)

```

```

a <- 0.7
b <- 3.8
Xce1 <- Xce2 <- Xce3 <- rep(0,m)
Xre1 <- Xre2 <- Xre3 <- rep(0,m)
for(i in 1:m)
{
  Z1 <- rlnorm(Ncal[i],xi1,sigma1)
  Z2 <- rlnorm(Ncal[i],xi2,sigma2)
  Z3 <- rlnorm(Ncal[i],xi3,sigma3)
  Xre1[i] <- sum(pmin(pmax(Z1-a,0),b))
  Xre2[i] <- sum(pmin(pmax(Z2-a,0),b))
  Xre3[i] <- sum(pmin(pmax(Z3-a,0),b))

  Xce1[i] <- sum(Z1)-Xre1[i]
  Xce2[i] <- sum(Z2)-Xre2[i]

```

```

    Xce3[i] <- sum(Z3)-Xre3[i]
  }
  sort(Xce1)[m*yps]
  sort(Xce2)[m*yps]
  sort(Xce3)[m*yps]

  sort(Xre1)[m*yps]
  sort(Xre2)[m*yps]
  sort(Xre3)[m*yps]

  pi_re1 <- mean(Xre1)
  pi_re2 <- mean(Xre2)
  pi_re3 <- mean(Xre3)

## Part II

# a)
m <- 1e5
rg <- 0.05
r <- 0.03
T <- 1
sig1 <- 0.2
sig2 <- 0.25
sig3 <- 0.3

#Monte Carlo
eps <- rnorm(m)
R1 <- exp(r*T-sig1^2*T/2+sig1*sqrt(T)*eps)-1
R2 <- exp(r*T-sig2^2*T/2+sig2*sqrt(T)*eps)-1
R3 <- exp(r*T-sig3^2*T/2+sig3*sqrt(T)*eps)-1
pi11 <- exp(-r*T)*mean(pmax(rg-R1,0))
pi22 <- exp(-r*T)*mean(pmax(rg-R2,0))
pi33 <- exp(-r*T)*mean(pmax(rg-R3,0))
pim <- c(pi11,pi22,pi33)

# b)

```

```

rc1 <- 0.06
rc2 <- 0.1
rc3 <- 0.15
rc4 <- 0.3
rc5 <- 0.4
rc <- c(rc1,rc2,rc3,rc4,rc5)

#Monte carlo
R <- cbind(R1,R2,R3)
X1 <-X2 <-X3 <-X4 <-X5 <- matrix(NA,m,3)
for(j in 1:3){
  X1[,j]<- pmin(pmax(rg-R[,j],0),rc[1]-R[,j])
  X2[,j]<- pmin(pmax(rg-R[,j],0),rc[2]-R[,j])
  X3[,j]<- pmin(pmax(rg-R[,j],0),rc[3]-R[,j])
  X4[,j]<- pmin(pmax(rg-R[,j],0),rc[4]-R[,j])
  X5[,j]<- pmin(pmax(rg-R[,j],0),rc[5]-R[,j])
  X <- cbind(X1,X2,X3,X4,X5)
  picli <- exp(-r*T)*apply(X,2,mean)
  picli.matrix <- matrix(picli,nrow=5,ncol=3,byrow=TRUE)
}

round(picli.matrix,4)

```