Exam in STK3505/4505: Answers

Problem 1

a

Simulation of Z using the inversion method requires an expression for the inverse cdf $F^{-1}(u)$:

$$F(z) = 1 - \frac{1}{\left(1 + \left(\frac{z}{\beta}\right)^{\theta}\right)^{\alpha}} = u$$

$$z = \beta\left((1-u)^{-1/\alpha} - 1\right)^{1/\theta} = F^{-1}(u).$$

Simulation algorithm:

```
Input: α, θ, β
Draw U* ~ U(0,1)
Return Z* = β((1 − U*)^(−1/α) − 1)^(1/θ)    % or β((U*)^(−1/α) − 1)^(1/θ)
```

Input: α, θ, β
Draw $U^* \sim U(0,1)$
Return $Z^* = \beta\left((1-U^*)^{-1/\alpha} - 1\right)^{1/\theta}$ % or $\beta\left((U^*)^{-1/\alpha} - 1\right)^{1/\theta}$

b

Simulation algorithm for $\mathcal{X}$:

Input: $\lambda, \alpha, \beta, m$
for i=1,...,m **do**
    Draw $\mathcal{N}^* \sim Poisson(\lambda)$
    $\mathcal{X}_i^* \leftarrow 0$
    for j=1,...,$\mathcal{N}^*$ **do**
        Draw $Z^* \sim Burr(\alpha, \theta, \beta)$
        $\mathcal{X}_i^* \leftarrow \mathcal{X}_i^* + Z^*$
    end for
end for
Return $\mathcal{X}_1^*, \ldots, \mathcal{X}_m^*$.

Estimate of the mean: $\bar{\mathcal{X}}^* = \frac{1}{m}\sum_{i=1}^{m} \mathcal{X}_i^*$.
Estimate of the standard deviation: $s_{\mathcal{X}}^* = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(\mathcal{X}_i^* - \bar{\mathcal{X}}^*\right)^2}$.
The $100 \cdot \epsilon\%$ reserve q_ϵ is given by

$$\mathrm{P}(\mathcal{X} \leq q_\epsilon) = \epsilon,$$

and estimated by $q^*_\epsilon = \mathcal{X}^*_{(m\epsilon)}$, where $\mathcal{X}^*_{(1)} \leq \ldots \leq \mathcal{X}^*_{(m)}$.

c

$\mathrm{E}(\mathcal{X}) = \lambda\xi(\alpha, \beta, \theta) = 72.6.$
$\mathrm{sd}(\mathcal{X}) = \sqrt{\lambda\,(\sigma^2(\alpha, \beta, \theta) + (\xi(\alpha, \beta, \theta))^2)} = 17.0.$

d

The 95% and 99% reserves are 101.9 and 118.2, respectively.

e

Simulations of cedent net portfolio payoffs $\mathcal{X}^{ce,*}_1, \ldots, \mathcal{X}^{ce,*}_m$ are obtained by replacing line 7 by
$7: Z^{ce,*} \leftarrow Z^* - \max(Z^* - a, 0)$
$8: \mathcal{X}^{ce,*}_i \leftarrow \mathcal{X}^{ce,*}_i + Z^{ce,*}$
to the simulations from **b**. The corresponding $100 \cdot \epsilon\%$ cedent net reserve is estimated by $q^{ce,*}_\epsilon = \mathcal{X}^*_{(m\epsilon)}$.

f

The cedent net 95% and 99% reserves are 74.9 and 83.4, respectively. Obviously, the reserves become lower when part of the responsibility is transfered on a reinsurer. Since there is no upper bound on this responsibility, but merely a retention limit, the reduction in the reserves is quite large, especially out in the tail.

g

$\mathcal{X}^{re} = \sum_{i=1}^{\mathcal{N}} Z^{re}_i = b\sum_{i=1}^{\mathcal{N}} Z_i = b\mathcal{X}$
$\mathcal{X}^{ce} = \mathcal{X} - \mathcal{X}^{re} = (1-b)\mathcal{X}$
Pure reinsurance premium: $\pi^{pu,re} = \mathrm{E}(\mathcal{X}^{re}) = b\mathrm{E}(\mathcal{X}) = 16.2.$
$100 \cdot \epsilon\%$ cedent net reserve:

$$\mathrm{P}(\mathcal{X}^{ce} \leq q^{ce}_\epsilon) = \mathrm{P}(\mathcal{X} \leq \frac{q^{ce}_\epsilon}{1-b}) = \epsilon = \mathrm{P}(\mathcal{X} \leq q_\epsilon)$$
$$q^{ce}_\epsilon = (1-b)q_\epsilon$$

The cedent net 95% and 99% reserves are now 79.2 and 91.8, respectively.

h

The pure reinsurance premium is the same for the two contracts, but the stoploss contract from **e** gives lower risk and lower reserves than the proportional contract in **g**, and is therefore preferable.

Problem 2

a

$$\pi_{l_0} = \begin{cases} s\sum_{k=l_r-l_0}^{\infty} d^k \ {}_k p_{l_0}, & l_0 < l_r \\ s\sum_{k=0}^{\infty} d^k \ {}_k p_{l_0}, & l_0 \geq l_r \end{cases}.$$

b)

The longer the time between the start age l_0 and the age of retirement l_r, the more the pension is discounted, and the smaller the present value. Thus, π_{l_0} ia an increasing function of l_0, so the order is:

l_0	47	57	37
π_{l_0}	4.98	7.40	3.46

c)

Present value of payments: $\zeta \sum_{k=0}^{l_r-l_0-1} d^k \ {}_k p_{l_0}$.

d)

Equivalence means that the expected present value of the payments ζ should be equal to the expected present value of the pension π_{l_0}, so that:

$$\zeta = s\frac{\sum_{k=l_r-l_0}^{\infty} d^k \ {}_k p_{l_0}}{\sum_{k=0}^{l_r-l_0-1} d^k \ {}_k p_{l_0}}$$

Problem 3

a)

$R_k = \frac{S_k}{S_{k-1}} - 1 = \frac{e^{Y_k}}{e^{Y_{k-1}}} - 1 = e^{Y_k - Y_{k-1}} - 1 = e^{X_k} - 1.$

K-step return: $R_{0:K} = (1 + R_1) \cdot \ldots \cdot (1 + R_K) - 1.$

$R_{0:K} = (1 + R_1) \cdot \ldots \cdot (1 + R_K) - 1 = e^{X_1} \cdot \ldots \cdot e^{X_K} - 1 = e^{\sum_{j=1}^{K} X_j} - 1 = e^{K\xi + \sigma \sum_{j=1}^{K} \epsilon_j}.$

Let $\eta = \frac{1}{\sqrt{K}} \sum_{j=1}^{K} \epsilon_j$. Since $\epsilon_j \overset{iid}{\sim} N(0,1)$, η is also normally distributed with

$$\mathrm{E}(\eta) = \frac{1}{\sqrt{K}} \sum_{j=1}^{K} \mathrm{E}(\epsilon_j) = 0$$

$$\mathrm{Var}(\eta) = \frac{1}{K} \sum_{j=1}^{K} \mathrm{Var}(\epsilon_j) = 1.$$

Thus, $R_{0:K} = e^{K\xi + \sigma\sqrt{K}\eta} - 1$, with $\eta \sim N(0,1)$.

b)

Simulation algorithm for X:

Input: r, σ, v_0, r_g, K

$\begin{cases} R^* \leftarrow 0 \\ P^* \leftarrow 1 \end{cases}$

for k=1,...,K **do**

  Draw $\epsilon^* \sim N(0,1)$

  $R^* \leftarrow e^{r - \frac{1}{2}\sigma^2 + \sigma\epsilon^*} - 1$

  $P^* \leftarrow P^*(1 + R^*)$

end for

$R^*_{0:K} \leftarrow P^* - 1$

Return $X^* \leftarrow \max(r_g - R^*_{0:K}, 0)$.

The risk-neutral price is computed as $\pi^*(v_0) = \frac{e^{-rK}}{m} \sum_{j=1}^{m} X_i^*$, where $X_1^*, \ldots, X_m^*$ are generated using the above algorithm.

$\pi(v_0)$ can also be computed using Black-Scholes formula:

$$\pi(v_0) = \left((1 + r_g) e^{-rK} \Phi(a) - \Phi(a - \sigma\sqrt{K}) \right) v_0,$$

where $a = \frac{\log(1+r_g) - rK + \sigma^2 K/2}{\sigma\sqrt{K}}$.

c

$$\mathrm{E}(\mathcal{R}) = \xi$$

$$\mathrm{Var}(\mathcal{R}) = \frac{\sigma^2}{J}$$

$$\frac{\mathrm{sd}(\mathcal{R})}{\mathrm{E}(\mathcal{R})} = \frac{\sigma/\xi}{\sqrt{J}} \underset{J \to \infty}{\longrightarrow} 0.$$

d

$$\begin{aligned}
\mathrm{E}(\mathcal{R}|R_M) &= r + \beta(R_M - r) \\
\mathrm{Var}(\mathcal{R}|R_M) &= \frac{\sigma^2}{J} \\
\mathrm{E}(\mathcal{R}) &= \mathrm{E}(\mathrm{E}(\mathcal{R}|R_M)) = r + \beta(\xi_M - r) \\
\mathrm{Var}(\mathcal{R}) &= \mathrm{Var}(\mathrm{E}(\mathcal{R}|R_M)) + \mathrm{E}(\mathrm{Var}(\mathcal{R}|R_M)) = \beta^2\sigma_M^2 + \frac{\sigma^2}{J} \\
\frac{\mathrm{sd}(\mathcal{R})}{\mathrm{E}(\mathcal{R})} &= \frac{\sqrt{\beta^2\sigma_M^2 + \sigma^2/J}}{r + \beta(\xi_M - r)} \underset{J\to\infty}{\longrightarrow} \frac{\beta\sigma_M}{r + \beta(\xi_M - r)}.
\end{aligned}$$

When all stocks depend on a common factor R_M, the risk cannot be diversified away, as in **c**.