UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK3505/4505 — Problems and methods in Actuarial science

Day of examination: Friday December 15th 2017

Examination hours: 14.30 – 18.30

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Introduction

Problem 1 is on general insurance, Problem 2 on life insurance and Problem 3 on financial risk. All of them must be answered and all sub-problems count equally. Write program sketches either in pseudo code or as R commands. The mathematical definitions and expressions below (that you need not prove or justify) may help you solving some of the sub-problems.

Rules of double expectation and variance:

$$\mathrm{E}(Y) = \mathrm{E}(\mathrm{E}(Y|\boldsymbol{X}))$$
$$\mathrm{Var}(Y) = \mathrm{Var}(\mathrm{E}(Y|\boldsymbol{X})) + \mathrm{E}(\mathrm{Var}(Y|\boldsymbol{X})).$$

Mean and standard deviation of the log-normal distribution: If $Z \sim log - N(\xi, \sigma)$, then

$$\mathrm{E}(Z) = e^{\xi + \frac{1}{2}\sigma^2} \quad \text{and} \quad \mathrm{sd}(Z) = \mathrm{E}(Z)\sqrt{e^{\sigma^2} - 1}.$$

Mean, standard deviation and cumulative distribution function of the Pareto distribution: If $Z \sim Pareto(\alpha, \beta)$, then

$$\mathrm{E}(Z) = \frac{\beta}{\alpha - 1} \quad \text{and} \quad \mathrm{sd}(Z) = \mathrm{E}(Z)\sqrt{\frac{\alpha}{\alpha - 2}},$$

for $\alpha > 1$ and $\alpha > 2$, respectively, and

$$F(z) = 1 - \frac{1}{\left(1 + \frac{z}{\beta}\right)^{\alpha}}, \ z > 0.$$

Estimation of a probability density function: A Monte Carlo estimate of the pdf $f(x)$ of a random variable X based on the independent simulations $x_1^*, \ldots, x_m^*$ is

$$f^*(x) = \frac{1}{mh}\sum_{i=1}^{m} K\left(\frac{x - x_i^*}{h}\right),$$

(Continued on page 2.)

where $K(\cdot)$ is a kernel function with mean 0 that integrates to 1, for instance a Gaussian kernel, and h is the bandwidth parameter.

Problem 1 General insurance

Let $\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}$ be the total pay-out in a general insurance portfolio under standard assumptions, where $\mathcal{N} \sim Poisson(\lambda)$ is independent of the individual losses Z_i, which are independent, identically $Pareto(\alpha, \beta)$ distributed.
Assume that for a given set of historical losses $z_1, \ldots, z_n$, the sample mean and standard deviation are $\bar{z} = 2.67$ and $s = 3.77$, respectively.

a

Show that the moment estimates of α and β are $\hat{\alpha} = 4.01$ and $\hat{\beta} = 8.04$. What challenges may you run into when using the method of moments to estimate the parameters of the Pareto distribution?

b

Derive a sampler for the Pareto distribution using the inversion method.

c

Sketch a program for generating m samples of $\mathcal{X}$. Explain how to determine $E(\mathcal{X})$ and $sd(\mathcal{X})$, as well as the reserve from these simulations (remember to define what the reserve is).

The table below shows the expectation, standard deviation and a few percentiles of the distribution of $\mathcal{X}$ when $\lambda = 30$. The number of simulations was $100,000$:

$E(\mathcal{X})$	$sd(\mathcal{X})$	1%	5%	25%	50%	75%	95%	99%
80.1	25.4	33.8	44.2	62.2	77.1	94.7	125.5	153.0

d

Derive expressions for $E(\mathcal{X})$ and $sd(\mathcal{X})$ for the given parameters (**Hint**: use the rules of double expectation and variance). Compare the exact values you get for $E(\mathcal{X})$ and $sd(\mathcal{X})$ with the ones from the table. Are there any signs of errors in the simulation program used for computing the values in the table?

e

What are the 95% and 99% reserves of the portfolio?

Suppose the portfolio is reinsured through a contract where the reinsurer responsibility is

$$\mathcal{X}^{re} = \begin{cases} 0, & \mathcal{X} \leq a \\ \mathcal{X} - a, & a < \mathcal{X} \leq a + b \\ b, & \mathcal{X} > a + b \end{cases},$$

(Continued on page 3.)

where $a, b > 0$ are fixed by the contract.

f

How is the net responsibility of the cedent simulated?

The cedent net responsibility $\mathcal{X}^{ce}$ is summarized in the table below for $a = 90$, $b = 35$ and other parameters as before.

$E(\mathcal{X}^{ce})$	$sd(\mathcal{X}^{ce})$	1%	5%	25%	50%	75%	95%	99%
74.9	18.0	34.1	44.3	62.2	77.3	90.0	90.3	118.4

g

What are the 95% and 99% **cedent** net reserves? Comment on the differences from **e**. What is the **reinsurance** pure premium and what is the actual reinsurance premium when the loading is $\gamma = 0.8$?

Problem 2 Life insurance

Suppose a pension starts to run at age l_r, lasting until the individual dies, with an amount s paid out at the beginning of each period. The probability of an individual of age l living at least k periods longer is ${}_kp_l$, and the discount per period is d.

a

Write down an expression for the expected present value π_{l_0} of such a pension for an individual at age l_0 when $l_0 < l_r$ and when $l_0 \geq l_r$, where you explain the different factors of the expression.

Figure a shows π_{l_0} as a function of l_0 when $s = 1$, the age of retirement is $l_r = 67$,

$$\log({}_kp_l) = -\theta_0 k - \frac{\theta_1}{\theta_2}\left(e^{\theta_2 k} - 1\right)e^{\theta_2 l},$$

with $\theta_0 = 0.009$, $\theta_1 = 0.000046$ and $\theta_2 = 0.0908$ and the discount is $d = 1/(1 + r)$, with r varied between 0.02, 0.03, 0.04, 0.05 (each curve corresponds to one value of r).

b

Explain the shape of the curves and identify which curve corresponds to which value of r.

c

Assume that the pension is financed by fixed contributions ζ at the start of each year from age $l_0 < l_r$ up to age $l_r - 1$. What is the expected present value of all these payments? Explain the different factors of the expression.

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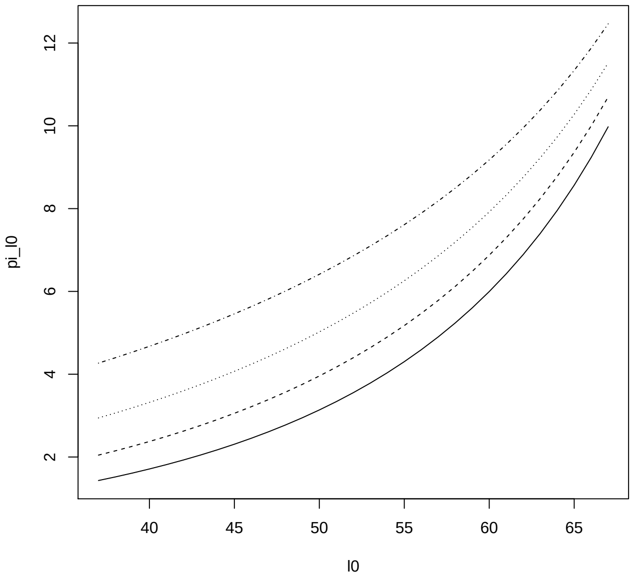

Figure 1: Premium π_{l_0} as a function of l_0 for each of the values of r.

d

What does it mean that ζ is determined by equivalence (explain in words)? Write down a mathematical expression for it when the contributions are made at the start of each period up to one period before retirement.

Problem 3 Financial risk

The random walk on logarithmic scale is a common model for equity. This means that the stock price S_k at time k is given by

$$S_k = e^{Y_k}$$
$$Y_k = Y_{k-1} + X_k, k = 1, 2, \dots,$$

where $X_k = \xi + \sigma\varepsilon_k$, $S_0 = v_0$ and the ε_ks are independent and $\sim N(0,1)$.

a

What is the K-step return $R_{0:K}$ and what is its probability distribution? (**Hint:** show first that $R_k = e^{X_k} - 1$).

b

Sketch a program for simulating $R_{0:K}$ and explain how you could use that to estimate its probability density function. Discuss the choice of bandwidth parameter h in general.

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An alternative is to put v_0 in a bank account with a floating interest rate r_k given by

$$r_k = re^{-\tau^2/(2(1-a^2))+X_k}$$
$$X_k = aX_{k-1} + \tau\varepsilon_k, k = 1, 2, \ldots,$$

where $|a| < 1$ and $x_0 = \log\left(\frac{r_0}{r}\right) + \frac{\tau^2}{2(1-a^2)}$. It may then be shown that (you do not have to prove this)

$$X_k = a^k x_0 + \tau \sum_{i=0}^{k-1} a^i \varepsilon_{k-i},$$

such that

$$\mathrm{E}(X_k|x_0) = a^k x_0$$
$$\mathrm{sd}(X_k|x_0) = \sqrt{\frac{1-a^{2k}}{1-a^2}}\tau.$$

c

Show that

$$\mathrm{E}(r_k|r_0) = re^{a^k x_0 - \tau^2 a^{2k}/(2(1-a^2))} \text{ and } \mathrm{sd}(r_k|r_0) = \mathrm{E}(r_k|r_0)\sqrt{e^{\tau^2(1-a^{2k})/(1-a^2)} - 1}$$

and use this to show that

$$\mathrm{E}(r_k|r_0) \underset{k\to\infty}{\longrightarrow} r \quad \text{and} \quad \mathrm{sd}(r_k|r_0) \underset{k\to\infty}{\longrightarrow} r\sqrt{e^{\tau^2/(1-a^2)} - 1}.$$

The table below shows the mean, standard deviation and lower and upper 5% quantiles of the value V of the investment v_0 in the stock (first row) and in the floating interest rate (second row) after K years, i.e. $(1+R_{0:K})v_0$ and $(1+r_{0:K})v_0$, respectively. The values are obtained with $v_0 = 1$, $\xi = 0.04$, $r = 0.04$, $\sigma = 0.25$, $a = 0.7$, $\tau = 0.25$ and $K = 10$.

$E(V)$	$sd(V)$	5%	95%
2.04	1.90	0.40	5.50
1.49	0.13	1.32	1.72

d

Comment on the differences between the two investments. What are the advantages and disadvantages?

END