Problem 1

a) $\pi^{pu} = \mathrm{E}(X)$.

b) $\pi = (1+\gamma)\pi^{pu}$, where π is the actual premium payed and γ the loading

c) Let $\mathcal{X} = X_1 + \ldots + X_J$ be the total payout from the portfolio. The $100 \cdot \epsilon\%$ reserve q_ϵ is given by

$$\mathrm{P}(\mathcal{X} \leq q_\epsilon) = \epsilon,$$

where ϵ is close to 1.

d) $\mathrm{E}(X_1 + \ldots + X_J) = \sum_{j=1}^{J} \mathrm{E}(X_j) = J\xi$ and $\mathrm{Var}(X_1 + \ldots + X_J) \overset{indep.}{=} \sum_{j=1}^{J} \mathrm{Var}(X_j) = J\sigma^2$. Hence,

$$\frac{\mathrm{sd}(X_1 + \ldots + X_J)}{\mathrm{E}(X_1 + \ldots + X_J)} = \frac{\sqrt{J}\sigma}{J\xi} = \frac{\sigma/\xi}{\sqrt{J}} \underset{J\to\infty}{\longrightarrow} 0.$$

Problem 2

a) Simulation of Z using the inversion algorithm requires an expression for the inverse cdf $F^{-1}(u)$:

$$\begin{aligned} F(z) = 1 - \frac{1}{(1+z/\beta)^\alpha} &= u \\ (1+z/\beta)^\alpha &= (1-u)^{-1} \\ z &= \beta\left((1-u)^{-1/\alpha} - 1\right) = F^{-1}(u). \end{aligned}$$

Simulation algorithm:

Input: α, β
Draw $U^* \sim U(0,1)$
Return $Z^* = \beta\left((1-U^*)^{-1/\alpha} - 1\right)$ % or $\beta\left((U^*)^{-1/\alpha} - 1\right)$

b) Simuleringsalgoritme for $\mathcal{X}$:

Input: $\lambda, \alpha, \beta, m$
for i=1,...,m **do**
    Draw $\mathcal{N}^* \sim Poisson(\lambda)$
    $\mathcal{X}_i^* \leftarrow 0$
    for j=1,...,$\mathcal{N}^*$ **do**

Draw $Z^* \sim Pareto(\alpha, \beta)$
$\mathcal{X}_i^* \leftarrow \mathcal{X}_i^* + Z^*$
end for
end for
Return $\mathcal{X}_1^*, \ldots, \mathcal{X}_m^*$.

c)

$$\begin{aligned}
\mathrm{E}(\mathcal{X}) &= \mathrm{E}(\mathrm{E}(\mathcal{X}|\mathcal{N})) \\
&= \mathrm{E}\left(\mathrm{E}\left(\sum_{i=1}^{\mathcal{N}} Z_i|\mathcal{N}\right)\right) \\
&= \mathrm{E}(\mathcal{N}\mathrm{E}(Z_i)) \\
&= \mathrm{E}(\mathcal{N})\mathrm{E}(Z_i) \\
&= \lambda\frac{\beta}{\alpha-1} = 25.
\end{aligned}$$

d) The 95% and 99% reserves are 48.8 and 64.2 respectively.

e) Simulations of reinsurer portfolio payoffs $\mathcal{X}_1^{re,*}, \ldots, \mathcal{X}_m^{re,*}$ are obtained by applying $\mathcal{X}_i^{re,*} \leftarrow \max(\mathcal{X}_i^* - a, 0)$ to the simulations from b).

f) The pure reinsurance premium is 3.15 and the reinsurance 95% and 99% reserves are 18.76 and 34.25 respectively.

Problem 3

a) Let Y be the number of years an individual is alive.

$$\begin{aligned}
{}_kp_{l_0} &= \mathrm{P}(Y > l_0 + k | Y > l_0) \\
&= \frac{\mathrm{P}(Y > l_0 + k)}{\mathrm{P}(Y > l_0)} \\
&= \frac{\mathrm{P}(Y > l_0 + k)}{\mathrm{P}(Y > l_0 + k - 1)} \cdot \frac{\mathrm{P}(Y > l_0 + k - 1)}{\mathrm{P}(Y > l_0 + k - 2)} \cdot \ldots \cdot \frac{\mathrm{P}(Y > l_0 + 1)}{\mathrm{P}(Y > l_0)} \\
&= \mathrm{P}(Y > l_0 + k | Y > l_0 + k - 1) \cdot \ldots \cdot \mathrm{P}(Y > l_0 + 1 | Y > l_0) \\
&= p_{l_{0+k-1}} p_{l_{0+k-2}} \cdot \ldots \cdot p_{l_0}
\end{aligned}$$

b) $V_0 = -\pi \sum_{i=0}^{K-1} d^i \;{}_ip_{l_0} + s \sum_{i=K}^{\infty} d^i \;{}_ip_{l_0}$.

c) $\pi = s\frac{\sum_{i=K}^{\infty} d^i \;{}_ip_{l_0}}{\sum_{i=0}^{K-1} d^i \;{}_ip_{l_0}}$

d) $V_k = \begin{cases} -\pi \sum_{i=k}^{K-1} d^{i-k} \;{}_ip_{l_0} + s \sum_{i=K}^{\infty} d^{i-k} \;{}_ip_{l_0}, & k < K \\ s \sum_{i=k}^{\infty} d^{i-k} \;{}_ip_{l_0}, & k \geq K \end{cases}$

e) The sum to be repaid is V_k.

Problem 4

a) $R_k = \frac{S_k}{S_{k-1}} - 1 = \frac{e^{Y_k}}{e^{Y_{k-1}}} - 1 = e^{Y_k - Y_{k-1}} - 1 = e^{X_k} - 1$.

b) $Y_k = Y_{k-1} + X_k = Y_{k-2} + X_{k-1} + X_k = \ldots = Y_0 + \sum_{j=1}^{k} X_j = \log(s_0) + \sum_{j=1}^{k} X_j$. Since $X_j \overset{iid}{\sim} N(\xi, \sigma)$, Y_k is also normally distributed with

$$\mathrm{E}(Y_k) = \log(s_0) + \mathrm{E}\left(\sum_{j=1}^{k} X_j\right) = \log(s_0) + \sum_{j=1}^{k} \mathrm{E}(X_j) = \log(s_0) + k\xi$$

$$\mathrm{sd}(Y_k) = \mathrm{sd}\left(\sum_{j=1}^{k} X_j\right) \overset{indep.}{=} \sqrt{\sum_{j=1}^{k} Var(X_j)} = \sqrt{k}\sigma.$$

Then, $S_k = e^{Y_k}$ must follow a log-normal distribution with

$$\mathrm{E}(S_k) = e^{\log(s_0) + k\xi + \frac{1}{2}k\sigma^2} = s_0 e^{k\xi + \frac{1}{2}k\sigma^2}.$$

c) Simulation algorithm for $S_1, \ldots, S_K$:

Input: ξ, σ, s_0, K
$Y^* \leftarrow \log(s_0)$
for k=1,...,K **do**
    Draw $X^* \sim N(\xi, \sigma)$
    $Y^* \leftarrow Y^* + X^*$
    $S^* \leftarrow e^{Y^*}$
end for
Return $S_1^*, \ldots, S_K^*$.

d) $\mathrm{E}(S_k) = s_0 e^{k\xi + \frac{1}{2}k\sigma^2} \approx 2.01$.

e) $R_{0:K} = (1+R_1)\cdot \ldots \cdot (1+R_K) - 1 = e^{X_1} \cdot \ldots \cdot e^{X_K} - 1 = e^{\sum_{j=1}^{K} X_j} - 1 = e^{Y_K - \log(s_0)} - 1 = \frac{S_K}{s_0} - 1$. Hence, $\mathrm{P}(R_{0:K} \leq r) = \mathrm{P}\left(\frac{S_K}{s_0} - 1 \leq r\right) = \mathrm{P}(S_K \leq s_0(r+1))$.

- Probability of losing on the investment: $\mathrm{P}(R_{0:K} < 0) = \mathrm{P}(S_K < s_0) = \mathrm{P}(S_K < 1)$, which is a little smaller than 0.25.
- Probability of doubling the investment: $\mathrm{P}(R_{0:K} \geq 1) = \mathrm{P}(S_K \geq 2s_0) = 1 - \mathrm{P}(S_K < 2)$, which is between 0.25 and 0.5.
- Probability of quadruplicating the investement: $\mathrm{P}(R_{0:K} \geq 3) = \mathrm{P}(S_K \geq 4s_0) = 1 - \mathrm{P}(S_K < 4)$, which is between 0.05 and and 0.25, but closer to 0.05.

Problem 5

a) Simulation algorithm for $B_1, \ldots, B_K$:

Input: $\xi, a, \tau, b_0, r_0, K$
$\begin{cases} B_0^* \leftarrow b_0 \\ Z^* \leftarrow \log\left(\frac{r_0}{\xi}\right) \end{cases}$
for k=1,…,K **do**
  Draw $\varepsilon^* \sim N(0,1)$
  $\begin{cases} Z^* \leftarrow aZ^* + \tau\varepsilon^* \\ r^* \leftarrow \xi e^{Z^*} \end{cases}$
  $B_k^* \leftarrow (1+r^*)B_{k-1}^*$
end for
Return $B_1^*, \ldots, B_K^*$.

b) Since S_k and B_k are independent, they can be simulated independently from the algorithms in 4 c) and 5 a), respectively, resulting in $\boldsymbol{S}_1^*, \ldots, \boldsymbol{S}_m^*$ and $\boldsymbol{B}_1^*, \ldots, \boldsymbol{B}_m^*$, with $\boldsymbol{S}_j^* = (S_{1j}^*, \ldots, S_{Kj}^*)^T$ and $\boldsymbol{B}_j^* = (B_{1j}^*, \ldots, B_{Kj}^*)^T$. Then, we obtain $\boldsymbol{V}_1^*, \ldots, \boldsymbol{V}_m^*$, with $\boldsymbol{V}_j^* = (V_{1j}^*, \ldots, V_{Kj}^*)^T$, from $\boldsymbol{V}_j^* = \boldsymbol{S}_j^* + \boldsymbol{B}_j^*$.

c) When half of the investment is placed in a cash account instead of stocks, the risk becomes lower, which is seen in the higher values for left tail quantiles and $\frac{\mathrm{sd}(V_K)}{\mathrm{E}(V_K)} \approx 0.41 < \frac{\mathrm{sd}(S_K)}{\mathrm{E}(S_K)} \approx 0.7$. On the other hand, the upside is also lower, which is seen in the lower expectation and right tail quantiles.