

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK3505/4505 — Answers

Day of examination: ?? . ?? . ????

This problem set consists of 7 pages.

Appendices: None

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Grade scale:

A: 74 – 80

B: 62 – 73

C: 46 – 61

D: 37 – 45

E: 32 – 36

F: 0 – 31

Problem 1 General insurance

1a

To find the moment matching estimates solve equations

$$8.36 = e^{\xi + \sigma^2/2},$$

and

$$\sqrt{4.48} = 8.36 \sqrt{e^{\sigma^2} - 1}$$

Theoretical moment should match observed moment. $\hat{\xi} = 2.092393$ and $\hat{\sigma} = 0.2492597$.

1b

$$\begin{aligned}\mathcal{L}(\xi, \sigma) &= \log \left(\prod_{i=1}^n f(z_i | \mu) \right) = \sum_{i=1}^n \log (f(z_i | \mu)) = \sum_{i=1}^n \log \left(\frac{1}{z_i \sigma \sqrt{2\pi}} e^{-\frac{(\ln(z_i) - \xi)^2}{2\sigma^2}} \right) = \\ &= - \sum_{i=1}^n \log(z_i) - \sum_{i=1}^n \log(\sigma) - \sum_{i=1}^n \log(\sqrt{2\pi}) - \sum_{i=1}^n \left(\frac{(\ln(z_i) - \xi)^2}{2\sigma^2} \right) = \\ &= - \sum_{i=1}^n \log(z_i) - \sum_{i=1}^n \log(\sigma) - \sum_{i=1}^n \log(\sqrt{2\pi}) - \sum_{i=1}^n \frac{(\ln(z_i))^2}{2\sigma^2} + \sum_{i=1}^n \frac{2\xi \ln(z_i)}{2\sigma^2} - n \frac{\xi^2}{2\sigma^2}\end{aligned}$$

(Continued on page 2.)

$$\frac{\partial \mathcal{L}(\xi, \sigma)}{\partial \xi} = \sum_{i=1}^n \frac{2 \ln(z_i)}{2\sigma^2} - 2n \frac{\xi}{2\sigma^2}$$

$$\frac{\partial \mathcal{L}(\xi, \sigma)}{\partial \xi} = 0 \text{ when } \hat{\xi} = \frac{\sum_{i=1}^n z_i}{n}.$$

$$\frac{\partial \mathcal{L}(\xi, \sigma)}{\partial \sigma} = -n \frac{1}{\sigma} + 2 \sum_{i=1}^n \frac{(\ln(z_i))^2}{2\sigma^3} - 2 \sum_{i=1}^n \frac{2\xi \ln(z_i)}{2\sigma^3} + 2n \frac{\xi^2}{2\sigma^3}$$

$$\frac{\partial \mathcal{L}(\xi, \sigma)}{\partial \sigma} = 0 \text{ when } n = \sum_{i=1}^n \frac{(\ln(z_i) - \xi)^2}{\sigma^2} \text{ so } \hat{\sigma} = \sqrt{\sum_{i=1}^n \frac{(\ln(z_i) - \hat{\xi})^2}{n}}$$

1c

Simulation of lognormal Z using the inversion method requires the inverse of standard normal distribution function:

Simulation algorithm:

1. Input: ξ, σ
2. Draw $U^* \sim U(0, 1)$
3. Return $Z^* = e^{\xi + \sigma \Phi^{-1}(U^*)}$

1d

Simulation algorithm for $\mathcal{X}$:

1. Input ξ, σ, λ, m
2. for $i = 1, \dots, m$ do
3. Draw $\mathcal{N}^* \sim \text{Poisson}(\lambda)$
4. $\mathcal{X}_i^* \leftarrow 0$
5. for $j = 1, \dots, \mathcal{N}^*$ do:
6. Draw $Z^* \sim \text{lognormal}(\xi, \sigma)$
7. $\mathcal{X}_i^* \leftarrow \mathcal{X}_i^* + Z^*$
8. end for
9. end for
10. Return $\mathcal{X}_1^*, \dots, \mathcal{X}_m^*$.

Estimate of the mean: $\overline{\mathcal{X}^*} = \frac{1}{m} \sum_{i=1}^m \mathcal{X}_i^*$.

Estimate of the standard deviation: $S_{\mathcal{X}^*}^* = \sqrt{\frac{1}{m-1} \sum_{i=1}^m (\mathcal{X}_i^* - \overline{\mathcal{X}^*})^2}$

The $100 * \epsilon\%$ reserve r_ϵ is given by $P(\mathcal{X} \leq q_\epsilon) = \epsilon$ and estimated by $q_\epsilon^* = \mathcal{X}_{(m\epsilon)}^*$, where $\mathcal{X}_{(1)}^* \leq \dots \leq \mathcal{X}_{(m)}^*$.

(Continued on page 3.)

1e

$$E(\mathcal{X}^*|\mathcal{N}) = E\left(\sum_{i=1}^{\mathcal{N}} Z_i\right) = \sum_{i=1}^{\mathcal{N}} E(Z_i) = \sum_{i=1}^{\mathcal{N}} e^{\tilde{\zeta} + \sigma^2/2} = \mathcal{N} e^{\tilde{\zeta} + \sigma^2/2}$$

$$E(\mathcal{X}^*) = E(E(\mathcal{X}^*|\mathcal{N})) = E(\mathcal{N} e^{\tilde{\zeta} + \sigma^2/2}) = \lambda e^{\tilde{\zeta} + \sigma^2/2} = 83.6$$

$$\text{var}(\mathcal{X}^*|\mathcal{N}) = \text{var}\left(\sum_{i=1}^{\mathcal{N}} Z_i\right) = \sum_{i=1}^{\mathcal{N}} \text{var}(Z_i) = \mathcal{N} \left(e^{\tilde{\zeta} + \sigma^2/2}\right)^2 (e^{\sigma^2} - 1)$$

$$\text{var}(\mathcal{X}^*) = \text{var}(E(\mathcal{X}^*|\mathcal{N})) + E(\text{var}(\mathcal{X}^*|\mathcal{N})) = \text{var}\left(\mathcal{N} e^{\tilde{\zeta} + \sigma^2/2}\right) + E\left(\mathcal{N} \left(e^{\tilde{\zeta} + \sigma^2/2}\right)^2 (e^{\sigma^2} - 1)\right)$$

$$= 8.36^2 \lambda + 4.48 \lambda = 698.9 + 44.8 = 743.7$$

$$sd(\mathcal{X}^*) = 27.27$$

1f

Simulations of reinsurer net portfolio payoffs $\mathcal{X}_1^{re,*}, \dots, \mathcal{X}_m^{re,*}$ are obtained by applying:

$$Z_i^{re,*} = \min(\max(Z_i^* - a, 0), b)$$

to the simulations of Z^* from **c**. To find the pure premium of the reinsurer $\pi^{pu,re}$ we need the expectation of the portfolio loss which is the mean of simulated $\mathcal{X}^{*,re}$

1g

$$\pi^{re} = (1 + \gamma) \mathcal{X}^{*,re}$$

1h

Normal with parameters $(\tilde{\zeta}, \frac{\tilde{\zeta}_\sigma}{G})$

1i

$$E(Y|G) = \tilde{\zeta}, \text{ so } E(E(Y|G)) = \tilde{\zeta}.$$

$$\text{var}(Y|G) = \frac{\tilde{\zeta}_\sigma^2}{G}, \text{ so } \text{var}(Y) = E\left(\frac{\tilde{\zeta}_\sigma^2}{G}\right) + \text{var}(\tilde{\zeta}) = \tilde{\zeta}_\sigma^2 E\left(\frac{1}{G}\right) = \tilde{\zeta}_\sigma^2 \frac{\alpha}{\alpha-1}.$$

(Continued on page 4.)

Problem 2 Life insurance

2a

Probability of surviving from age 65 to age 68 is equal to the probability of surviving 3 years given that the individual is of age 65

$${}_3p_{65} = {}_1p_{65} \cdot {}_1p_{66} \cdot {}_1p_{67}$$

${}_1p_{65}$ corresponds to $(1 - \text{probability of dying between ages 65 and 66})$ from the life table. That means that

$${}_3p_{65} = (1 - 0.013) \cdot (1 - 0.014) \cdot (1 - 0.015) = 0.9585843$$

2b

Simulation algorithm:

1. Input: $l_0 = 65, K = 2, J_M, J_W = 500, p_{1,M}, p_{2,M}, p_{1,W}, p_{2,W}$

Draw the number of people surviving the first year:

2. Draw $\mathcal{N}_{1,M}^* \sim \text{Binomial}(J_M, p_{1,M})$

3. Draw $\mathcal{N}_{1,W}^* \sim \text{Binomial}(J_W, p_{1,W})$

Draw the number of people surviving the second year:

4. Draw $\mathcal{N}_{2,M}^* \sim \text{Binomial}(\mathcal{N}_{1,M}^*, p_{2,M})$

5. Draw $\mathcal{N}_{2,W}^* \sim \text{Binomial}(\mathcal{N}_{1,W}^*, p_{2,W})$

6. $\mathcal{N}_{1,M}^* + \mathcal{N}_{1,W}^* + \mathcal{N}_{2,M}^* + \mathcal{N}_{2,W}^*$

Women will receive more payments, because their probabilities of survival are higher.

2c

	To new state			
From	Active	Disabled	Dead	Row sum
Active	$p_l(1 - p_l(i a))$	$p_l p_l(i a)$	$1 - p_l$	1
Disabled	$p_l(p_l(a i))$	$p_l(1 - p_l(a i))$	$1 - p_l$	1
Dead	0	0	1	1

(Continued on page 5.)

2d

state S : 1 - active, 2 - disabled, 3 - dead

1. Input: probabilities of survival, probabilities of transitions
2. $N^* = 0$
3. $S = 1$ (person is active)
4. for $k = 1, 2, 3$:
 5. if $S = 3$: (if person is dead)
 6. end for
 7. if $S = 2$: (if person is disabled)
 8. Draw $U^* \sim Uniform$
 9. If $U^* < p_{65+k-1}$: (if person survived)
 10. $N^* = N^* + 1$
 11. else $S = 3$ (person died)
 12. if $S = 1$: (if person is active)
 13. Draw $U^* \sim Uniform$
 14. If $U^* < p_{65+k-1} * 0.95$: (if person survived and stayed active)
 15. continue
 16. If $U^* < p_{65+k-1}$: (if person survived and became disabled)
 17. $S = 2, N^* = N^* + 1$
 18. If $U^* \geq p_{65+k-1}$: (if person died)
 19. $S = 3$
20. Return N^*

Problem 3 Financial risk**3a**

The premium for call option in terms of single asset is

$$\pi(v_0) = e^{-rT} E_Q(\max(R - r_g, 0))v_0,$$

(Continued on page 6.)

3b

The premium for call option in terms of single asset is

$$\pi(v_0) = e^{-rT} E_Q(\max(R - r_g, 0))v_0,$$

where $R = e^{\xi_q T + \sigma\sqrt{T}\epsilon} - 1$ for $\epsilon \sim N(0, 1)$.

There is positive payoff if $R > r_g$ or equivalently if $\epsilon > a$ where

$$a = \frac{\log(1 + r_g) - \xi_q T}{\sigma\sqrt{T}}$$

and the option premium becomes

$$\pi(v_0) = e^{-rT} \left(\int_a^\infty (e^{\xi_q T + \sigma\sqrt{T}x} - 1 - r_g) \phi(x) dx \right).$$

Splitting the integrand gives

$$\pi(v_0) = e^{-rT} \left(-(1 + r_g) \int_a^\infty \phi(x) dx + e^{\xi_q T} \int_a^\infty e^{\sigma\sqrt{T}x} \phi(x) dx \right)$$

where the second integral on the right is

$$\int_a^\infty e^{\sigma\sqrt{T}x} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} dx = e^{\sigma^2 \frac{T}{2}} \int_a^\infty (2\pi)^{-\frac{1}{2}} e^{-\frac{(x - \sigma\sqrt{T})^2}{2}} dx.$$

If $\Phi(x) = \int_{-\infty}^x \phi(y) dy$ is the Gaussian integral then $1 - \Phi(x) = \Phi(-x)$ and

$$\pi(v_0) = e^{-rT} \left(-(1 + r_g)(\Phi(-a)) + e^{\xi_q T + \sigma^2 \frac{T}{2}} (\Phi(-a + \sigma\sqrt{T})) \right) v_0$$

and inserting $\xi_q = r - \frac{\sigma^2}{2}$ into the expression and into a yields the Black-Scholes formula.

3c

1. Input: parameters $r_g, r, \sigma, T, m, w_1, w_2, \rho$ etc
2. for $i = 1, \dots, m$ do:
3. Draw $\epsilon_1, \epsilon_2 \sim N(0, 1)$ iid
4. $\epsilon_2 \leftarrow \rho\epsilon_1 + \sqrt{1 - \rho^2}\epsilon_2$
5. $R_{i,1}^* \leftarrow e^{rT - \sigma_1^2 \frac{T}{2} + \sigma_1 \sqrt{T}\epsilon_1} - 1$
6. $R_{i,2}^* \leftarrow e^{rT - \sigma_2^2 \frac{T}{2} + \sigma_2 \sqrt{T}\epsilon_2} - 1$
7. $X_i^* \leftarrow \max(R_{i,1}^* - R_{i,2}^*, 0)$
8. end for
9. return $\pi^* = \frac{e^{-rT}}{m} \sum_{i=1}^m X_i^*$

The higher the correlation - the higher the price

(Continued on page 7.)

3d

$$R_k = \frac{S_k}{S_{k-1}} - 1 = \frac{e^{Y_k}}{e^{Y_{k-1}}} - 1 = e^{Y_k - Y_{k-1}} - 1 = e^{X_k} - 1$$

K-step return: $R_{0:K} = (1 + R_1) \cdot \dots \cdot (1 + R_K) - 1$

$$R_{0:K} = (1 + R_1) \cdot \dots \cdot (1 + R_K) - 1 = e^{X_1} \cdot \dots \cdot e^{X_K} - 1 = e^{\sum_{j=1}^K X_j} - 1 = e^{K\xi + \sigma \sum_{j=1}^K \epsilon_j}$$

Let $\eta = \frac{1}{\sqrt{K}} \sum_{j=1}^K \epsilon_j$. Since $\epsilon_j \sim N(0, 1)$ and iid, η is also normally distributed with

$$E(\eta) = \frac{1}{\sqrt{K}} \sum_{j=1}^K E(\epsilon_j) = 0$$

$$\text{var}(\epsilon_j) = \frac{1}{K} \sum_{j=1}^K \text{var}(\epsilon_j) = 1.$$

So $R_{0:K} = e^{K\xi + \sigma\sqrt{K}\eta} - 1$, with $\eta \sim N(0, 1)$ and $R_{0:K} \sim \text{lognormal}(K\xi, \sigma\sqrt{K})$

3e

1. Input: parameters $\xi, \sigma, K, B, H, S_0, m$
2. $Y_0 = \ln(S_0)$
3. for $i = 1, \dots, m$ do:
4. For $k = 1, 2, 3$:
5. Draw $\epsilon^* \sim N(0, 1)$
6. $Y_k^* = Y_{k-1}^* + \xi + \sigma\epsilon^*$
7. $S_k^* = e^{Y_k^*}$
8. If $\max_{k=0,1,2,3} S_k < B$:
9. $X^* = \max(H - S_K, 0)$
10. Else:
11. $X_i^* = 0$
12. end for
13. end for
14. return $\pi^* = \frac{e^{-rT}}{m} \sum_{i=1}^m X_i^*$

END