

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK3505/4505 — Problems and methods in Actuarial science

Day of examination: 19.12.2022

Examination hours: 15:00–19:00

This problem set consists of 14 pages.

Appendices: None

Permitted aids: Aproved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Grade	points
A	[73, 81]
B	[61, 73)
C	[47, 61)
D	[40, 47)
E	[32.5, 40)
F	[0, 32.5)

Table 1: Grade scale

Problem 1 Non-life insurance

Let $\mathcal{X} = Z_1 + \cdots + Z_{\mathcal{N}}$ be the sum of claims for a non-life insurance product with 1 year coverage time under standard assumptions, i.e. $\mathcal{N} \sim \text{Poisson}(\lambda)$ is independent of the individual claims Z_i and the sequence $\{Z_i\}_i$ is i.i.d. ("compound Poisson model"). The individual claim amount Z_i is distributed according to the exponential distribution with parameter $\mu > 0$, with the following density, expected value and variance

$$\mathbb{E}(Z_i) = \frac{1}{\mu},$$

$$\text{VAR}(Z_i) = \frac{1}{\mu^2},$$

$$f(x) = \mu e^{-\mu x} 1_{[0, \infty)}(x)$$

1a 3pt

Find the moment matching estimator for μ . You can use the sample mean $\bar{X}_n$ as a moment.

(Continued on page 2.)

Solution: To find a moment matching estimator of μ we need to solve the following equation wrt μ

$$\bar{X}_n = \frac{1}{\mu}$$

Therefore $\hat{\mu}_{MM} = \frac{1}{\bar{X}_n}$

1b 6pt

Find the expression for the maximum likelihood estimator of μ . Remember to check that the critical point is a global maximum.

Solution:

Likelihood function has a form:

$$\mathcal{L}(\mu, x_1, \dots, x_n) = \prod_{i=1}^n \mu e^{-\mu x_i}$$

Compute the loglikelihood function

$$\begin{aligned} l(\mu, x_1, \dots, x_n) &= \sum_{i=1}^n [\ln \mu - \mu x_i] = \\ &= n \ln \mu - \mu \sum_{i=1}^n x_i \end{aligned}$$

To find a maximum likelihood estimator we look for a critical point and need to check that it is a maximum. Derivate therefore loglikelihood w.r.t. the parameter

$$\frac{\partial}{\partial \mu} l(\mu, x_1, \dots, x_n) = n \frac{1}{\mu} - \sum_{i=1}^n x_i$$

Therefore

$$\frac{\partial}{\partial \mu} l(\mu, x_1, \dots, x_n) = 0 \Leftrightarrow n \frac{1}{\mu} - \sum_{i=1}^n x_i = 0 \Leftrightarrow \mu = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{X}_n}$$

We must check if the critical point is a maximum therefore compute the second-order derivative

$$\frac{\partial^2}{\partial \mu^2} l(\mu, x_1, \dots, x_n) = -n \frac{1}{\mu^2}$$

The second derivative is negative and therefore the critical point is a global maximum

Therefore

$$\mu_{ML} = \frac{1}{\bar{X}_n}$$

(Continued on page 3.)

1c 3pt

Sketch a program for a sampler for the exponential distribution using the inversion method. Note that you need to find the inverse of cumulative distribution function and use it in the inversion method.

Solution:

Compute first CDF

$$\begin{aligned} F(u) &= \int_{-\infty}^u f(s)ds = \int_{-\infty}^u \mu e^{-\mu s} 1_{[0,\infty)}(s)ds = \int_0^u \mu e^{-\mu s} ds 1_{[0,\infty)}(u) \\ &= -e^{-\mu s} \Big|_{s=0}^{s=u} 1_{[0,\infty)}(u) = [1 - e^{-\mu u}] 1_{[0,\infty)}(u) \end{aligned}$$

Assume that $u > 0$ and find the inverse of CDF:

$$F(u) = 1 - e^{-\mu u} \Leftrightarrow e^{-\mu u} = 1 - F(u) \Leftrightarrow u = \frac{-\ln(1 - F(u))}{\mu}$$

Therefore

$$F^{-1}(y) = \frac{\ln(1 - y)}{\mu}$$

Sampler therefore has the following form

1. Inputs: μ
2. Draw $U \sim \text{Uniform}(0, 1)$
3. Set $X = \frac{-\ln(1-U)}{\mu}$ or alternatively $X = \frac{-\ln(U)}{\mu}$
4. Return X

1d 6pt

Sketch a program for generating m samples of $\mathcal{X}$. Explain how to determine $\mathbb{E}(\mathcal{X})$ and $\mathbb{VAR}(\mathcal{X})$.

Solution:

1. Inputs: μ, λ, m
2. For j in 1 to m do
 - (a) Set $X_j = 0$
 - (b) Draw $N \sim \text{Poiss}(\lambda)$,
 - (c) for i in 1 to N do
 - i. draw $Z_i \sim \text{Exp}(\mu)$
 - ii. $X_j = X_j + Z_i$
3. Return $X_1, \dots, X_m$

(Continued on page 4.)

In order to determine $\mathbb{E}(\mathcal{X})$ and $\mathbb{VAR}(\mathcal{X})$ one need to compute the mean and variance from the simulated data

$$\mathbb{E}(\mathcal{X}) \approx \frac{1}{m} \sum_{j=1}^m X_j$$

and

$$\mathbb{VAR}(\mathcal{X}) \approx \frac{1}{m-1} \sum_{j=1}^m \left(X_j - \frac{1}{m} \sum_{j=1}^m X_j \right)^2$$

1e 2pt

Give the expressions for $\mathbb{E}(\mathcal{X})$ and $\mathbb{VAR}(\mathcal{X})$. You do not need to prove it.

Solution:

$$\mathbb{E}(\mathcal{X}) = \mathbb{E}(N)\mathbb{E}(Z_i) = \lambda \frac{1}{\mu}$$

and

$$\mathbb{VAR}(\mathcal{X}) = \mathbb{E}(N)\mathbb{VAR}(Z_i) + (\mathbb{E}Z_i)^2\mathbb{VAR}(N) = \lambda \left(\frac{1}{\mu^2} + \frac{1}{\mu^2} \right) = \frac{2\lambda}{\mu^2}$$

Suppose the portfolio is reinsured through the stop loss reinsurance where the reinsurer recoverable per portfolio are given by a function

$$\mathcal{X}^{re} = \begin{cases} 0 & \text{if } \mathcal{X} \leq a, \\ \mathcal{X} - a & \text{if } a < \mathcal{X} \end{cases}$$

where $a > 0$ are fixed by the contract.

1f 2pt

How would you use algorithm from point 1d to calculate the pure premium for the reinsurer for that contract?

Solution: The pure premium for the reinsurer is computed as expected value of recoverables. We can use Monte Carlo method to approximate it. The algorithm from 1d is the same however, we need to replace line 3 with the following:

Return $\max(X_1 - a, 0), \dots, \max(X_m - a, 0)$

1g 4pt

The pure premium $\pi^{pu} = 300$. The insurer set a premium using a loading $\gamma = 0.3$ and sold 25 policies on June 1st, 20 policies on July 1st and 30 policies on August 1st, all with one-off premium paid in advance. Today is September 1st. The policy's coverage time is 1 year.

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1. Calculate premium paid by each customer. (1 pt)
2. Calculate the liability for unearned exposure per September 1st according to the accounting rules for all policies sold. Tip: you need to calculate unearned premium per September 1st for each policies. (1.5 pt)
3. Calculate the Premium Provision under Solvency II per September 1st. (Tip: pure premium = expected total amount of claims for whole coverage time of 12 months and you are interested in expected claims in the remaining period of the coverage) (1.5pt)

You can assume that the acquisition costs, expenses and the discounting rate are all 0. Assume that the year has 360 days and each month has 30 days. Exposure is earned linearly.

Solution:

Premium paid by each customer is equal to $\pi = \pi^{pu}(1 + \gamma) = 300 * 1.3 = 390$

Liability for unearned exposure is equal to unearned premium (minus acquisition costs). Unearned premium for policies sold in June 1st is equal to 9 month/12 months * overall premium for 12 months * number of policies and therefore it is equal to $9/12 * 390 * 25 = 7312.5$. Similarly, unearned premium for policies sold in July 1st is equal to $10/12 * 390 * 20 = 6500$ and for policies sold in August 1st: $11/12 * 390 * 30 = 10725$. Therefore liability for unearned exposure is equal to $7312.5 + 6500 + 10725 = 24537.5$.

Solvency II Premium provision is equal to discounted sum of future expected claims and expenses minus discounted sum of future premiums. Expenses are assumed to be equal 0, premiums are paid upfront and discounting rate is equal 0 and therefore the Premium Provision in this case is equal to future expected claims. Note that pure premium is equal to expected claims in the 12 months period. Since exposure is earned linearly, we compute expected future claims as a remaining coverage time divided by overall coverage time times pure premium. Premium provisions for policies sold in June 1st is therefore equal to 9 month/12 months * overall pure premium for 12 months * number of policies and therefore it is equal to $9/12 * 300 * 25 = 5625$. Similarly, premium provisions for policies sold in July 1st is equal to $10/12 * 300 * 20 = 5000$ and for policies sold in August 1st: $11/12 * 300 * 30 = 8250$. Therefore the total Premium provision is equal to $5625 + 5000 + 8250 = 18875$.

1h 3pt

The same product was offered a year ago. Pure premium for that product was 360 a year ago. The insurer sold 200 policies, all on September 1st of the previous year. Today is September 1st of the current year and the insurer paid out in total 30 000. Calculate the claim provision held by the insurer using the two following methods:

(Continued on page 6.)

1. a naive method , which assumes that the total claim amounts paid out is going to be equal to the expected value of the total claim amounts used for calculation of the pure premium. (1.5 pt)
2. a payment pattern method, assuming that 40% of total claim amount is (on average) paid out within the first year of coverage. (1.5 pt)

You can assume that expenses and discounting interest rate are 0.

Solution:

The claim provision is a discounted sum of expected future payments for claims and expenses for the earned exposure. In this case, all 200 policies are fully earned, there are no expenses and the discounting rate is equal to zero. Therefore the claim provision is equal to expected to expected total amount of claims minus claim amounts which have been already paid out.

For the naive method, the expected total claim amounts is exactly the same as expected value of the total claim amounts used to calculate pure premium. Therefore the expected total claim amounts is $360 \cdot 200 = 72000$. The Claim provision is therefore equal to $72000 - 30000 = 42000$.

For the payment pattern method we update the estimate of the expected total claim amounts taking into consideration what has been paid out and how much on average the insurer pays out in that period. The new estimate is equal to the amount which is paid out (30 000) divided by 40% and is therefore equal to 75000. The Claim provision calculated using this method is therefore equal to $75000 - 30000 = 45000$.

Problem 2 Life insurance - two state model

Let Y be the length of life for a representative individual in years (i.e. Y takes only values in natural numbers). For $l \in \mathbb{N}$ we define the probability of dying at age l as $q_l = \mathbb{P}(Y = l + 1 | Y > l)$. The table below presents the values of q_l

Age: l	q_l
65	0.01
66	0.0105
67	0.011
68	0.012
69	0.013

Table 2: Probability of dying at age l q_l

2a 3pt

For $k \in \mathbb{N}$ define ${}_k p_l$ as ${}_k p_l = \mathbb{P}(Y > l + k | Y > l)$. What is the interpretation of it? Calculate ${}_k p_l$ for $l = 65$ and $k = 1, 2, 3, 4, 5$. Tip. You

(Continued on page 7.)

do not need to prove the formula for ${}_k p_l$ as long as you explain the idea behind it.

Solution: ${}_k p_l = \mathbb{P}(Y > l + k | Y > l)$ is a probability of staying alive in next k years from year l . It can be computed using the formula that ${}_k p_l = (1 - q_l) * \dots * (1 - q_{l+k-1})$. The idea behind formula is that surviving k next years from age l is equivalent to survive 1 year from age l , then surviving 1 year from age $l+1$, and so on and on up to surviving 1 year from age $l+k-1$. Probability of surviving 1 year from age $l+i$ is then equal to $(1 - q_{l+i})$.

Therefore:

$${}_1 p_{65} = 1 - q_{65} = 1 - 0.01 = 0.99$$

$${}_2 p_{65} = (1 - q_{65})(1 - q_{66}) = (1 - 0.01)(1 - 0.0105) = 0.99 * 0.9895 = 0.9796$$

$$\begin{aligned} {}_3 p_{65} &= (1 - q_{65})(1 - q_{66})(1 - q_{67}) = (1 - 0.01)(1 - 0.0105)(1 - 0.011) \\ &= 0.99 * 0.9895 * 0.989 = 0.9688 \end{aligned}$$

$$\begin{aligned} {}_4 p_{65} &= (1 - q_{65})(1 - q_{66})(1 - q_{67})(1 - q_{68}) = (1 - 0.01)(1 - 0.0105)(1 - 0.011)(1 - 0.012) \\ &= 0.99 * 0.9895 * 0.989 * 0.988 = 0.9572 \end{aligned}$$

$$\begin{aligned} {}_5 p_{65} &= (1 - q_{65})(1 - q_{66})(1 - q_{67})(1 - q_{68})(1 - q_{69}) \\ &= (1 - 0.01)(1 - 0.0105)(1 - 0.011)(1 - 0.012)(1 - 0.013) \\ &= 0.99 * 0.9895 * 0.989 * 0.988 * 0.987 = 0.9448 \end{aligned}$$

2b 2pt

For $k \in \mathbb{N}$ define ${}_k q_l$ as ${}_k q_l = \mathbb{P}(Y = l + k | Y > l)$. Argue for that ${}_k q_l = {}_{k-1} p_l * q_{l+k-1}$ and calculate it for $l = 65$ and $k = 1, 2, 3, 4, 5$. Again, a formal proof is not necessary as long as you can give the idea behind it

Solution: ${}_k q_l = \mathbb{P}(Y = l + k | Y > l)$ is a probability of dying in the k -th year from age l . It can be computed using the formula that ${}_k q_l = (1 - q_l) * \dots * (1 - q_{l+k-2}) q_{l+k-1}$. The idea behind formula is that dying in the k -th year from age l is equivalent to survive 1 year from age l , then surviving 1 year from age $l+1$, and so on and on up to surviving 1 year from age $l+k-2$ and dying in the age $l+k-1$.

Therefore:

$${}_1 q_{65} = q_{65} = 0.01$$

$${}_2 q_{65} = (1 - q_{65}) q_{66} = (1 - 0.01) 0.0105 = 0.99 * 0.0105 = 0.010395$$

$$\begin{aligned} {}_3 q_{65} &= (1 - q_{65})(1 - q_{66}) q_{67} = (1 - 0.01)(1 - 0.0105) 0.011 \\ &= 0.99 * 0.9895 * 0.011 = 0.010776 \end{aligned}$$

$$\begin{aligned} {}_4 q_{65} &= (1 - q_{65})(1 - q_{66})(1 - q_{67}) q_{68} = (1 - 0.01)(1 - 0.0105)(1 - 0.011) 0.012 \\ &= 0.99 * 0.9895 * 0.989 * 0.012 = 0.011626 \end{aligned}$$

$$\begin{aligned} {}_5 q_{65} &= (1 - q_{65})(1 - q_{66})(1 - q_{67})(1 - q_{68}) q_{69} \\ &= (1 - 0.01)(1 - 0.0105)(1 - 0.011)(1 - 0.012) 0.013 \\ &= 0.99 * 0.9895 * 0.989 * 0.988 * 0.013 = 0.012444 \end{aligned}$$

(Continued on page 8.)

2c 8pt

A term insurance pays a fixed amount if a person dies before a certain age l_r . Assume that a person who has just turned 65 has signed a term insurance which is going to pay 1 000 000 kr if that individual dies before turning 67. The payout of the benefit to the family occurs on the first "birthday" after death. The premium for this product is paid as a one-time premium paid when signing the contract, i.e. when the insured turns 65. The interest rate is constant $r = 5\%$.

1. Explain the equivalence principle for calculating this premium. (2 pt)
2. Calculate this one time premium using the equivalence principle and the transition probabilities from point 2b. (4 pt)
3. Would the premium increase or decrease if the the interest would fall to e.g. 2%? Explain why (you do not need to prove it or calculate it, but you should be able to reason it). (2pt)

Solution:

The premium calculated according to the equivalence principle is a premium π such that the expected present value of all the cashflows is equal to 0. It means that the premium covers on average the expected benefits from the policy and in this sense it is a fair premium.

The expected present value of the cashflows in this case is as follows:

$$V_0 = -\pi * \frac{1}{(1+r)^0} * 1 + 1000000 \sum_{k=1}^2 {}^kq_{65} \frac{1}{(1+r)^k}$$

Therefore the premium according to the equivalence principle is equal to

$$\begin{aligned} \pi &= 1000000 \sum_{k=1}^2 {}^kq_{65} \frac{1}{(1+r)^k} \\ &= 1000000 \left(\frac{{}^1q_{65}}{1.05} + \frac{{}^2q_{65}}{1.05^2} \right) \\ &= 1000000 \left(\frac{0.01}{1.05} + \frac{0.010395}{1.05^2} \right) = 18952 \end{aligned}$$

The premium would increase if the interest rate gets lower and this is due to the fact that the discounting effect of the future cashflows is going to be lower (i.e. we discount them less).

Problem 3 Life insurance - three state model

Consider now a three state insurance modelled for an active individual aged $l_0 = 65$. The Markov chain $(X_j)_{j=0,1,\dots}$ represents the state of the

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individual in j -years and can take three possible states: a - individual is alive and active, i - individual is alive and disabled, d - individual is dead.

$p_l(i|a)$ is the probability of getting disabled (**conditional on surviving**) for an active individual of age l , whereas $p_l(a|i)$ is the probability of recovery from disability (**conditional on surviving**) for a disabled individual of age l . Assume that abovementioned probability of recovery is $p_l(a|i) = 0.1$ (conditional on surviving) and probability of getting disabled is $p_l(i|a) = 0.05$ (conditional on surviving) for all $l = 65, \dots, 70$.

Regarding the survival probability, the insurance company has noticed that the disabled people live shorter than the active people. People who are alive and active has higher probability of survival of age l and we set it as $p_l^a = 0.99$, whereas disabled people have lower probability of survival $p_l^i = 0.98$ ($l = 65, \dots, 70$).

3a 6pt

Let $p_{xy}(j) = P(X_j = y | X_{j-1} = x)$ for $x, y \in \{a, i, d\}$. Let $\mathbb{P}(j) = [p_{xy}(j)]_{x,y \in \{a,i,d\}}$ be a transition probability matrix for this Markov chain. Compute this transition probability matrix for $j = 1, 2, 3, 4, 5$. Tip: You should note that the transition probability matrix for this problem does not depend on j as all of the probabilities in the problem are constant for $l = 65, \dots, 70$.

Solution:

$$\begin{aligned} \mathbb{P}(j) &= [p_{xy}(j)]_{x,y \in \{a,i,d\}} = \begin{matrix} & \begin{matrix} a & i & d \end{matrix} \\ \begin{matrix} a \\ i \\ d \end{matrix} & \begin{pmatrix} p_{aa}(j) & p_{ai}(j) & p_{ad}(j) \\ p_{ia}(j) & p_{ii}(j) & p_{id}(j) \\ p_{da}(j) & p_{di}(j) & p_{dd}(j) \end{pmatrix} \end{matrix} = \\ &= \begin{matrix} & \begin{matrix} a & i & d \end{matrix} \\ \begin{matrix} a \\ i \\ d \end{matrix} & \begin{pmatrix} p_{l_0+j-1}^a(1 - p_{l_0+j-1}(i|a)) & p_{l_0+j-1}^a p_{l_0+j}(i|a) & 1 - p_{l_0+j-1}^a \\ p_{l_0+j-1}^i p_{l_0+j-1}(a|i) & p_{l_0+j-1}^i(1 - p_{l_0+j-1}(a|i)) & 1 - p_{l_0+j-1}^i \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \\ &= \begin{matrix} & \begin{matrix} a & i & d \end{matrix} \\ \begin{matrix} a \\ i \\ d \end{matrix} & \begin{pmatrix} 0.99 * 0.95 & 0.99 * 0.05 & 0.01 \\ 0.98 * 0.1 & 0.98 * 0.9 & 0.02 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \\ &= \begin{matrix} & \begin{matrix} a & i & d \end{matrix} \\ \begin{matrix} a \\ i \\ d \end{matrix} & \begin{pmatrix} 0.9405 & 0.0495 & 0.01 \\ 0.098 & 0.882 & 0.02 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \end{aligned}$$

Note: if the student has a small mistake and writes " j " instead of " $j-1$ " (e.g. $p_{l_0+j}^a(1 - p_{l_0+j}(i|a))$), accept the solution without subtracting points.

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3b 8pt

Calculate the probability $P(X_4 = i | X_0 = a)$, i.e. probability that a person in four years is disabled under the condition that right now that person is active and at age $l_0 = 65$. Tip: $p_l(a|i) = 0.1 > 0$ and therefore you need to consider a possibility that an insured gets disabled, recovers and gets disabled again.

Solution: Due to the possible transitions for the Markov chain we can represent the event $\{X_4 = i, X_0 = a\}$ as

$$\begin{aligned} \{X_0 = a, X_4 = i\} = & \{X_0 = a, X_1 = a, X_2 = a, X_3 = a, X_4 = i\} \cup \\ & \{X_0 = a, X_1 = a, X_2 = a, X_3 = i, X_4 = i\} \cup \\ & \{X_0 = a, X_1 = a, X_2 = i, X_3 = i, X_4 = i\} \cup \\ & \{X_0 = a, X_1 = i, X_2 = i, X_3 = i, X_4 = i\} \cup \\ & \{X_0 = a, X_1 = i, X_2 = a, X_3 = i, X_4 = i\} \cup \\ & \{X_0 = a, X_1 = i, X_2 = a, X_3 = a, X_4 = i\} \cup \\ & \{X_0 = a, X_1 = i, X_2 = i, X_3 = a, X_4 = i\} \cup \\ & \{X_0 = a, X_1 = a, X_2 = i, X_3 = a, X_4 = i\} \end{aligned}$$

By conditioning and Markov property we get then that

$$\begin{aligned} P(X_4 = i | X_0 = a) = & p_{aa}(1)p_{aa}(2)p_{aa}(3)p_{ai}(4) \\ & + p_{aa}(1)p_{aa}(2)p_{ai}(3)p_{ii}(4) \\ & + p_{aa}(1)p_{ai}(2)p_{ii}(3)p_{ii}(4) \\ & + p_{ai}(1)p_{ii}(2)p_{ii}(3)p_{ii}(4) \\ & + p_{ai}(1)p_{ia}(2)p_{ai}(3)p_{ii}(4) \\ & + p_{ai}(1)p_{ia}(2)p_{aa}(3)p_{ai}(4) \\ & + p_{ai}(1)p_{ii}(2)p_{ia}(3)p_{ai}(4) \\ & + p_{aa}(1)p_{ai}(2)p_{ia}(3)p_{ai}(4) \\ = & (0.9405)^3(0.0495) + \\ & + (0.9405)^2(0.0495)(0.882) \\ & + (0.9405)^1(0.0495)(0.882)^2 \\ & + (0.0495)(0.882)^3 \\ & + (0.0495)^2(0.098)(0.882) \\ & + (0.9405)(0.0495)^2(0.098) \\ & + (0.0495)^2(0.098)(0.882) \\ & + (0.9405)(0.0495)^2(0.098) = 0.150852388 \end{aligned}$$

The same result we can get if we calculate $\mathbb{P}(1)\mathbb{P}(2)\mathbb{P}(3)\mathbb{P}(4) = \mathbb{P}(1)^4$ and take (a,i)-element of that matrix.

Regarding the grading: if a student uses the first method: subtract 1 pt for general calculation errors and subtract 1 pt for each trajectory of a

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Markov chain which a student forgot about. If a student multiplies the transition matrix give 5 pt for stating the solution and 3 pt for correct calculations (subtracting points depending on the mistakes).

Problem 4 Financial risk

Consider a derivative with maturity T written on the asset price S_T such the pay-off of the derivative is equal $g(S_T)$ for some function g . Assume that $S_T = s_0 \exp(\mu T + \sigma \sqrt{T} \epsilon)$, where μ and σ are parameters of the model and $\epsilon \sim N(0, 1)$ under a probability $\mathbb{P}$. Assume that r is a continuously compounded risk-free interest rate.

4a 2pt

Express a price of this derivative via expected present value of the pay-off. Under which probability you need to calculate the expected present value? (You do not need to compute the expectation here).

Solution: The value of the derivative at time 0 can be computed as

$$V_0 = e^{-rT} \mathbb{E}^Q[g(S_T)]$$

where Q is a risk neutral probability.

4b 4pt

Under the risk-neutral probability Q , the discounted present value of S_T must be equal to s_0 in order for the model not introduce the arbitrage (i.e. profit without risk). Assume that under risk-neutral probability Q S_T has the following form $S_T = s_0 \exp(\mu_Q T + \sigma \sqrt{T} \epsilon)$, where $\epsilon \sim N(0, 1)$ under a probability Q . What must be the value of μ_Q so that the abovementioned condition for arbitragefreeness is satisfied? Tip: For a log-normal distribution with parameters ξ and σ the expected value can be expressed as $\exp(\xi + \sigma^2/2)$.

Solution: The discounted present value of S_T is equal to

$$\begin{aligned} V_0 &= e^{-rT} \mathbb{E}^Q[S_T] = e^{-rT} \mathbb{E}^Q[s_0 \exp(\mu_Q T + \sigma \sqrt{T} \epsilon)] \\ &= s_0 e^{-rT + \mu_Q T + \frac{\sigma^2 T}{2}} \end{aligned}$$

Therefore $V_0 = s_0 \Leftrightarrow \mu_Q = \left(r - \frac{\sigma^2}{2}\right)$.

That tells us that S_T under the probability Q has the following distribution $S_T = s_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma \sqrt{T} \epsilon\right)$ where $\epsilon \sim N(0, 1)$ under a probability Q

(Continued on page 12.)

4c 3pt

Sketch an algorithm which would approximate the price of this derivative via Monte Carlo simulation.

Solution:

1. Inputs: σ, r, T, m, g
2. For j in 1 to m do
 - (a) Draw $\epsilon_j \sim N(0, 1)$,
 - (b) Set $X_j = g(s_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T}\epsilon_j\right))$
3. Return $V_0^{MC} = e^{-rT} \frac{1}{m} \sum_{j=1}^m X_j = \frac{e^{-rT}}{m} \sum_{j=1}^m g\left(s_0 e^{\left(r - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T}\epsilon_j}\right)$

4d 10pt

Consider a put option with maturity T and strike K written on the asset price S_T with the pay-off $X_P = \max(K - S_T, 0)$. Alternatively, this call option can be expressed as option on return R with pay-off $X_P = \max(r_g - R, 0)s_0$, where r_g is the minimum return, R is a return over time-period T . The relation between strike and the minimum return is then $K = s_0(1 + r_g)$. Note that $R = S_T/s_0 - 1$.

Present the proof of the Black-Scholes formula for the value of the put option at time $t = 0$ (i.e. derive it from scratch).

Solution:

Let $\phi(x)$ be a standard normal density and $\Phi(x)$ be a cumulative distribution function of a standard normal distribution. We compute:

$$\begin{aligned} V_0 &= e^{-rT} \mathbb{E}^Q[\max(K - S_T, 0)] \\ &= e^{-rT} \mathbb{E}^Q \left[\max \left(K - s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma\sqrt{T}\epsilon \right), 0 \right) \right] \\ &= e^{-rT} \int_{-\infty}^{\infty} \left[\max \left(K - s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma\sqrt{T}x \right), 0 \right) \phi(x) \right] dx \end{aligned}$$

Note that if x satisfies following $K - s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma\sqrt{T}x \right) < 0$ then the expression under the integral is 0. Transforming this we get the following condition for integral to be 0:

$$x > a = \frac{\ln \left(\frac{K}{s_0} \right) - \left(r - \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}}$$

Returning to our integral we get

(Continued on page 13.)

$$\begin{aligned}
V_0 &= e^{-rT} \int_{-\infty}^{\infty} \left[\max \left(K - s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right), 0 \right) \right] \phi(x) dx \\
&= e^{-rT} \int_{-\infty}^a \left[K - s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) \right] \phi(x) dx \\
&= e^{-rT} K \int_{-\infty}^a \phi(x) dx \\
&\quad - e^{-rT} \int_{-\infty}^a \left[s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) \right] \phi(x) dx =: I_1 - I_2
\end{aligned}$$

Focus first on I_1 .

$$I_1 = e^{-rT} K \int_{-\infty}^a \phi(x) dx = K e^{-rT} \Phi(a)$$

We also have

$$\begin{aligned}
I_2 &= e^{-rT} \int_{-\infty}^a \left[s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) \right] \phi(x) dx \\
&= e^{-rT} \int_{-\infty}^a \left[s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= s_0 e^{-rT + \left(r - \frac{\sigma^2}{2} \right) T} \int_{-\infty}^a e^{\sigma \sqrt{T} x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= s_0 e^{-\left(\frac{\sigma^2}{2} \right) T} \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 - 2\sigma \sqrt{T} x}{2}} dx \\
&= s_0 e^{-\frac{\sigma^2 T}{2}} \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 - 2\sigma \sqrt{T} x + \sigma^2 T - \sigma^2 T}{2}} dx \\
&= s_0 e^{-\frac{\sigma^2 T}{2}} e^{-\frac{\sigma^2 T}{2}} \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \sigma \sqrt{T})^2}{2}} dx \\
&= s_0 \int_{-\infty}^{a - \sigma \sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = s_0 \Phi(a - \sigma \sqrt{T})
\end{aligned}$$

Summarizing we get

$$V_0 = K e^{-rT} \Phi(a) - s_0 \Phi(a - \sigma \sqrt{T})$$

4e 6pt

A put-call parity is the relation between a price put V_0^P and call option V_0^C on the same underlying instrument S , with the same maturity T and the same strike K (alternatively the same guaranteed return r_g):

$$V_0^C = V_0^P + s_0 - K e^{-rT}$$

(Continued on page 14.)

Prove it. (Tip: use point 4a)

Solution: Introduce notation $X_P = \max(K - S_T, 0)$, $X_C = \max(S_T - K, 0)$. Note that $X_C - X_P = S_T - K$ Using point 4a as well as non-arbitrage condition in 4b one gets:

$$\mathbb{E}^Q[e^{-rT}(X_C - X_P)] = \mathbb{E}^Q[e^{-rT}(S_T - K)] = s_0 - Ke^{-rT}$$

On the other hand

$$\mathbb{E}^Q[e^{-rT}(X_C - X_P)] = \mathbb{E}^Q[e^{-rT}X_C] - \mathbb{E}^Q[e^{-rT}(X_P)] = V_0^C - V_0^P$$

Comparing both equations one gets put-call parity

END