

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK3505/4505 — Problems and methods in Actuarial science

Day of examination: 06.12.2023

Examination hours: 15:00–19:00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 is on non-life insurance, Problem 2 is on life insurance with two states, Problem 3 is on life insurance with 3 states and Problem 4 on financial risk. Points for each sub-problem are given next to the number of the sub-problem. Write program sketches either in pseudo code.

Problem 1 Non-life insurance

Let $\mathcal{X} = Z_1 + \dots + Z_{\mathcal{N}}$ be the sum of claims for a non-life insurance product with 1 year coverage time under standard assumptions, i.e. $\mathcal{N} \sim \text{Poisson}(\lambda)$ is independent of the individual claims Z_i and the sequence $\{Z_i\}_i$ is i.i.d. ("compound Poisson model"). The individual claim amount Z_i is distributed according to the lognormal distribution with parameter μ and $\sigma > 0$, with the following density, expected value and variance

$$\mathbb{E}(Z_i) = \exp\left(\mu + \frac{\sigma^2}{2}\right),$$

$$\text{VAR}(Z_i) = \left[\exp(\sigma^2) - 1\right] \exp\left(2\mu + \sigma^2\right),$$

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right)$$

1a 4pt

Find the moment matching estimator for the parameters of the individual claim amount distribution μ and σ . You can use the sample mean $\bar{Z}_m$ and sample standard deviation s_m as moments.

(Continued on page 2.)

1b 7pt

Find the expression for the maximum likelihood estimator of μ and σ . You do not need to check that the critical point is a global maximum.

1c 5pt

Sketch a program for generating m samples of $\mathcal{X}$. Explain how to approximate $\mathbb{E}(\mathcal{X})$ and $\mathbb{VAR}(\mathcal{X})$ from this simulated sample.

1d 6pt

State the rule of double expectation and of double variance (without proving them). Use them to derive the explicit formulas for $\mathbb{E}(\mathcal{X})$ and $\mathbb{VAR}(\mathcal{X})$.

Suppose the portfolio is reinsured through the proportional reinsurance where the reinsurance recoverable per portfolio are given by a function

$$\mathcal{X}^{re} = a\mathcal{X}$$

where $0 < a \leq 1$ are fixed by the contract.

1e 2pt

How would you use algorithm from point 1c to calculate the pure premium for the reinsurer for that contract?

1f 4pt

The pure premium $\pi^{pu} = 500$. The insurer set a premium using a loading $\gamma = 0.2$ and sold 30 policies on January 1st, 40 policies on February 1st and 35 policies on March 1st, all with one-off premium paid in advance. Today is April 1st. The policy's coverage time is 1 year.

1. Calculate premium paid by each customer. (1 pt)
2. Calculate the liability for remaining coverage per April 1st according to the accounting rules for all policies sold. Tip: you need to calculate unearned premium per April 1st for each policies. (1.5 pt)
3. Calculate the Premium Provision under Solvency II per April 1st. (Tip: pure premium = expected total amount of claims for whole coverage time of 12 months and you are interested in expected claims in the remaining period of the coverage) (1.5pt)

You can assume that the acquisition costs, expenses and the discounting rate are all 0. Assume that the year has 360 days and each month has 30 days. Exposure is earned linearly.

(Continued on page 3.)

1g 3pt

The same product was offered a year ago. Pure premium for that product was 450 a year ago. The insurer sold 30 policies, all on April 1st of the previous year. Today is April 1st of the current year and the insurer paid out in total 6 000. Calculate the claim provision held by the insurer using the two following methods:

1. a naive method, which assumes that the total claim amounts paid out is going to be equal to the expected value of the total claim amounts used for calculation of the pure premium. (1.5 pt)
2. a payment pattern method, assuming that 50% of total claim amount is (on average) paid out within the first year of coverage. (1.5 pt)

You can assume that expenses and discounting interest rate are 0.

Problem 2 Life insurance - two state model

Let Y be the length of life for a representative individual in years (i.e. Y takes only values in natural numbers). For $l \in \mathbb{N}$ we define the probability of dying at age l as $q_l = \mathbb{P}(Y = l + 1 | Y > l)$. The table below presents the values of q_l

Age: l	q_l
60	0.005
61	0.01
62	0.0105
63	0.011
64	0.012

Table 1: Probability of dying at age l q_l

2a 3pt

For $k \in \mathbb{N}$ define ${}_k p_l$ as ${}_k p_l = \mathbb{P}(Y > l + k | Y > l)$. What is the interpretation of it? Calculate ${}_k p_l$ for $l = 60$ and $k = 1, 2, 3, 4, 5$. Tip. You do not need to prove the formula for ${}_k p_l$ as long as you explain the idea behind it.

2b 9pt

A pure endowment insurance pays a fixed amount if a person reaches a certain retirement age l_r . Assume that a person who has just turned 60 has signed a pure endowment insurance which is going to pay 1 000 000 kr if that individual reaches a retirement age $l_r = 65$. The premium for this product is paid as a one-time premium paid when signing the contract, i.e. when the insured turns 60. The interest rate is constant $r = 5\%$.

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1. Explain the equivalence principle for calculating this premium. (2 pt)
2. Calculate this one time premium using the equivalence principle and the transition probabilities from point 2a. (4 pt)
3. What is a mathematical reserve for that policy? (1 pt)
4. Would the premium increase or decrease if the interest would increase to 6%? Explain why (you do not need to prove it or calculate it, but you should be able to reason it). (1pt)
5. Assume that the insurer has re-evaluated mortality assumptions and q_l doubles as a result. Would that lead to a premium increase or decrease? Explain why (you do not need to prove it or calculate it, but you should be able to reason it). (1pt)

Problem 3 Life insurance - three state model

Consider now a three state insurance modelled for an active individual aged $l_0 = 60$. The Markov chain $(X_j)_{j=0,1,\dots}$ represents the state of the individual in j -years and can take three possible states: a - individual is alive and active, i - individual is alive and disabled, d - individual is dead.

Let $p_{xy}(j) = P(X_j = y | X_{j-1} = x)$ for $x, y \in \{a, i, d\}$. Let $\mathbb{P}(j) = [p_{xy}(j)]_{x,y \in \{a,i,d\}}$ be a transition probability matrix for this Markov chain. Assume that for $j = 1, 2, 3, 4$ the matrix $\mathbb{P}(j)$ has a form:

$$\mathbb{P}(j) = [p_{xy}(j)]_{x,y \in \{a,i,d\}} = \begin{matrix} & \begin{matrix} a & i & d \end{matrix} \\ \begin{matrix} a \\ i \\ d \end{matrix} & \begin{pmatrix} p_{aa}(j) & p_{ai}(j) & p_{ad}(j) \\ p_{ia}(j) & p_{ii}(j) & p_{id}(j) \\ p_{da}(j) & p_{di}(j) & p_{dd}(j) \end{pmatrix} \end{matrix} =$$

$$= \begin{matrix} & \begin{matrix} a & i & d \end{matrix} \\ \begin{matrix} a \\ i \\ d \end{matrix} & \begin{pmatrix} 0.95 & 0.04 & 0.01 \\ 0 & 0.99 & 0.01 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

3a 2pt

Draw a graph of this Markov chain and state which transitions are not possible.

3b 6pt

For $k = 1, 2, 3, 4$, calculate the probabilities $P(X_k = i | X_0 = a)$, i.e. probability that a person in k years is disabled under the condition that right now that person is active and at age $l_0 = 60$.

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3c 8pt

An active and alive individual aged $l_0 = 60$ has signed a disability insurance which is going to pay a recurring benefit 1000 if the individual is disabled and did not reach the retirement age $l_r = 65$. The payment is made at the beginning of each year (e.g. if insured is getting disabled while between age 60 and 61, he gets the first payment when turning 61). Calculate the mathematical reserve for this policy under the assumption that $r = 5\%$.

Problem 4 Financial risk

Consider a derivative with maturity T written on the asset price S_T such the pay-off of the derivative is equal $g(S_T)$ for some function g . Assume that $S_T = s_0 \exp(\mu T + \sigma \sqrt{T} \epsilon)$, where μ and σ are parameters of the model and $\epsilon \sim N(0, 1)$ under a probability $\mathbb{P}$. Assume that r is a continuously compounded risk-free interest rate.

4a 5pt

Assume that you have two instruments: an asset with a market price S_0 at time 0 and a value S_T at time T as well as a derivative with a market price V_0 at time 0 and which replicates exactly the value of the asset S_T at time T (i.e. cashflow of derivative $g(S_T)$ is exactly equal to S_T with probability 1). Assume that $V_0 > S_0$. Construct an arbitrage, i.e. a portfolio of an asset, a derivative and money on a bank account which has value 0 at time 0 and a positive value with probability 1 at time T . You can assume that money on a bank account yields a positive interest rate r . Assume also that you are able both to buy a derivative and to issue it/sell it (i.e. someone else buys it from you for V_0 and you will pay the payoff at maturity).

4b 3pt

Under the risk-neutral probability $\mathbb{Q}$, the discounted present value of S_T must be equal to s_0 in order for the model not introduce the arbitrage (i.e. profit without risk). Assume that S_T has the following form $S_T = s_0 \exp(\mu_Q T + \sigma \sqrt{T} \epsilon)$ under risk-neutral probability $\mathbb{Q}$, where $\epsilon \sim N(0, 1)$ under a probability $\mathbb{Q}$. What must be the value of μ_Q so that the abovementioned condition for arbitragefreeness is satisfied? Tip: For a log-normal distribution with parameters ξ and σ the expected value can be expressed as $\exp(\xi + \sigma^2/2)$.

4c 3pt

Sketch an algorithm which would approximate the price of this derivative $\mathbb{E}^{\mathbb{Q}}[g(S_T)]e^{-rT}$ via Monte Carlo simulation.

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4d 10pt

Consider a call option with maturity T and strike K written on the asset price S_T with the pay-off $X_C = \max(S_T - K, 0)$. Alternatively, this call option can be expressed as option on return R with pay-off $X_C = \max(R - r_g, 0)s_0$, where r_g is the minimum return, R is a return over time-period T . The relation between strike and the minimum return is then $K = s_0(1 + r_g)$. Note that $R = S_T/s_0 - 1$.

Present the proof of the Black-Scholes formula for V_0^C , i.e. the value of the call option at time $t = 0$ (i.e. derive it from scratch).

4e 3pt

Delta is a derivative of the Black-Scholes option price with respect to the price of underlying instrument. It can be shown that delta for a put option can be expressed as:

$$\Delta = -\Phi(a - \sigma\sqrt{T}).$$

Use call-put parity $V_0^C = V_0^P + S_0 - Ke^{-rT}$ to calculate delta for a call option.

END