

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK3505/4505 — Introduction to Insurance Mathematics

Day of examination: 29.11.2024

Examination hours: 15:00–19:00

This problem set consists of 14 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Grade	points
A	[77.5, 86]
B	[64.5, 77.5)
C	[51.5, 64.5)
D	[43, 51.5)
E	[34.5, 43)
F	[0, 34.5)

Table 1: Grade scale

Problem 1 is on non-life insurance, Problem 2 is on life insurance with two states, Problem 3 is on life insurance with 3 states and Problem 4 on financial risk. Points for each sub-problem are given next to the number of the sub-problem. Write program sketches either in pseudo code.

Problem 1 Non-life insurance

Let $\mathcal{X} = Z_1 + \cdots + Z_{\mathcal{N}}$ be the sum of claims for a non-life insurance product with 1 year coverage time under standard assumptions, i.e. $\mathcal{N} \sim \text{Poisson}(\lambda)$ is independent of the individual claims Z_i and the sequence $\{Z_i\}_i$ is i.i.d. ("compound Poisson model"). The individual claim amount Z_i is distributed according to the exponential distribution with parameter μ , with the following density:

$$f(x) = \begin{cases} \mu \exp(-\mu x) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

1a 4pt

Derive the formula for the expected value of the exponential distribution. Find the moment matching estimator for the parameters of the individual

(Continued on page 2.)

claim amount distribution μ . You can use the sample mean $\bar{Z}_m$ as the moment.

Solution: If $Z \sim \text{Exp}(\mu)$ then

$$\begin{aligned}\mathbb{E}[Z] &= \int_0^\infty x\mu \exp(-\mu x)dx = \text{integration by parts} = \\ &= [-x \exp(-\mu x)]_0^\infty + \int_0^\infty \exp(-\mu x)dx = 0 - 0 - \frac{1}{\mu} [\exp(-\mu x)]_0^\infty \\ &= -\frac{1}{\mu} [0 - 1] = \frac{1}{\mu}.\end{aligned}$$

Then moment estimator solves the equation: $\bar{Z}_m = \frac{1}{\mu}$, i.e. $\hat{\mu}_{MM} = \frac{1}{\bar{Z}_m}$.

1b 5pt

Find the expression for the maximum likelihood estimator of μ . Remember to check that the critical point is a global maximum.

Solution:

Likelihood function has a form:

$$\mathcal{L}(\mu, z_1, \dots, z_n) = \prod_{i=1}^n \mu e^{-\mu z_i}$$

Compute the loglikelihood function

$$\begin{aligned}l(\mu, z_1, \dots, z_n) &= \sum_{i=1}^n [\ln \mu - \mu z_i] = \\ &= n \ln \mu - \mu \sum_{i=1}^n z_i\end{aligned}$$

To find a maximum likelihood estimator we look for a critical point and need to check that it is a maximum. Derivate therefore loglikelihood w.r.t. the parameter

$$\frac{\partial}{\partial \mu} l(\mu, z_1, \dots, z_n) = n \frac{1}{\mu} - \sum_{i=1}^n z_i$$

Therefore

$$\frac{\partial}{\partial \mu} l(\mu, z_1, \dots, z_n) = 0 \Leftrightarrow n \frac{1}{\mu} - \sum_{i=1}^n z_i = 0 \Leftrightarrow \mu = \frac{n}{\sum_{i=1}^n z_i} = \frac{1}{\bar{Z}_n}$$

We must check if the critical point is a maximum therefore compute the second-order derivative

$$\frac{\partial^2}{\partial \mu^2} l(\mu, z_1, \dots, z_n) = -n \frac{1}{\mu^2}$$

(Continued on page 3.)

The second derivative is negative and therefore the critical point is a global maximum

Therefore

$$\mu_{ML} = \frac{1}{\bar{Z}_n}$$

1c 3pt

Sketch a program for a sampler for the exponential distribution using the inversion method. Note that you need to compute the cumulative distribution function, find its inverse function and use it in the inversion method.

Solution:

Compute first CDF

$$\begin{aligned} F(u) &= \int_{-\infty}^u f(s) ds = \int_{-\infty}^u \mu e^{-\mu s} 1_{[0, \infty)}(s) ds = \int_0^u \mu e^{-\mu s} ds 1_{[0, \infty)}(u) \\ &= -e^{-\mu s} \Big|_{s=0}^{s=u} 1_{[0, \infty)}(u) = [1 - e^{-\mu u}] 1_{[0, \infty)}(u) \end{aligned}$$

Assume that $u > 0$ and find the inverse of CDF:

$$F(u) = 1 - e^{-\mu u} \Leftrightarrow e^{-\mu u} = 1 - F(u) \Leftrightarrow u = \frac{-\ln(1 - F(u))}{\mu}$$

Therefore

$$F^{-1}(y) = -\frac{\ln(1 - y)}{\mu}$$

Sampler therefore has the following form

1. Inputs: μ
2. Draw $U \sim \text{Uniform}(0, 1)$
3. Set $X = \frac{-\ln(1-U)}{\mu}$ or alternatively $X = \frac{-\ln(U)}{\mu}$
4. Return X

1d 4pt

Sketch a program for generating m samples of $\mathcal{X}$. Explain how to approximate $\mathbb{E}(\mathcal{X})$ and $\text{VAR}(\mathcal{X})$ using Monte Carlo mean and variance estimators?

Solution:

1. Inputs: μ, λ, m
2. For j in 1 to m do
 - (a) Set $X_j = 0$

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- (b) Draw $N \sim \text{Poiss}(\lambda)$,
- (c) for i in 1 to N do
 - i. draw $Z_i \sim \text{Exp}(\mu)$
 - ii. $X_j = X_j + Z_i$
- 3. Return $X_1, \dots, X_m$

In order to approximate $\mathbb{E}(\mathcal{X})$ and $\mathbb{VAR}(\mathcal{X})$ one need to compute the mean and variance from the simulated data

$$\mathbb{E}(\mathcal{X}) \approx \frac{1}{m} \sum_{j=1}^m X_j$$

and

$$\mathbb{VAR}(\mathcal{X}) \approx \frac{1}{m-1} \sum_{j=1}^m \left(X_j - \frac{1}{m} \sum_{j=1}^m X_j \right)^2$$

1e 5pt

State the rule of double expectation and of double variance (without proving them). Use them to derive the explicit formulas for $\mathbb{E}(\mathcal{X})$ and $\mathbb{VAR}(\mathcal{X})$.

Solution: Rule of double expectation

$$\mathbb{E}(\mathbb{E}(\mathcal{X}|N)) = \mathbb{E}(\mathcal{X})$$

Rule of double variance

$$\mathbb{VAR}(\mathcal{X}) = \mathbb{VAR}(\mathbb{E}(\mathcal{X}|N)) + \mathbb{E}(\mathbb{VAR}(\mathcal{X}|N)).$$

Given that:

$$\mathbb{E}(\mathcal{X}|N) = NZ_1$$

and

$$\mathbb{VAR}(\mathcal{X}|N) = N\mathbb{VAR}(Z_1)$$

we get:

$$\mathbb{E}(\mathcal{X}) = \mathbb{E}(N)\mathbb{E}(Z_1) = \frac{\lambda}{\mu}$$

and

$$\mathbb{VAR}(\mathcal{X}) = \mathbb{E}(N)\mathbb{VAR}(Z_1) + (\mathbb{E}Z_1)^2\mathbb{VAR}(N) =$$

$$= \text{Optional, check the comment below: } \lambda \frac{1}{\mu^2} + \frac{1}{\mu^2} \lambda = \frac{2\lambda}{\mu^2}$$

Important. Formulas for $\mathbb{VAR}(Z_1)$ or $\mathbb{E}Z_1$ were not given in the exam set and students were told that they do not need to memorize the formulas

(Continued on page 5.)

for the moments. Therefore do not subtract points for not plugging in formulas for the moments or plugging in wrong formulas.

Suppose the portfolio is reinsured through the stop loss reinsurance where the reinsurance recoverable per portfolio are given by a function

$$\mathcal{X}^{re} = \max(\mathcal{X} - a * P, 0)$$

where a is fixed by the contract and P is gross earned premium (given).

1f 2pt

How would you use algorithm from point 1d to estimate the pure premium for the reinsurer for that contract? Use a and P as input.

Solution: The pure premium for the reinsurer is computed as expected value of recoverables. We can use Monte Carlo method to approximate it. The algorithm from 1c is the same however, we need to replace line 1 with following:

Inputs: μ, λ, m, a, P

and line 3 with the following:

Return $\max(X_1 - aP, 0), \dots, \max(X_m - aP, 0)$

1g 6pt

Development Year:	1	2	3	4
Accident year:				
1	1100	1200	1230	1250
2	1150	1220	1240	
3	1420	1600		
4	1200			

Table 2: Cumulative paid triangle

Use the cumulative paid triangle given above to calculate Chain Ladder development factors. Calculate the estimate of the ultimate for each accident year as well as the undiscounted reserves per each accident year.

Solution: Chain ladder development factors:

$$\hat{f}_1 = \frac{1200+1220+1600}{1100+1150+1420} = 1.095368$$

$$\hat{f}_2 = \frac{1230+1240}{1200+1220} = 1.020661$$

$$\hat{f}_3 = \frac{1250}{1230} = 1.01626$$

Ultimates:

$$U_1 = 1250$$

$$U_2 = 1240 * \hat{f}_3 = 1260.16$$

$$U_3 = 1600 * \hat{f}_3 \hat{f}_2 = 1659.61$$

(Continued on page 6.)

$$U_4 = 1200 * \hat{f}_3 \hat{f}_2 \hat{f}_1 = 1363.41$$

Reserves:

$$R_1 = U_1 - C_{14} = 1250 - 1250 = 0$$

$$R_2 = U_2 - C_{23} = 1260.16 - 1240 = 20.16$$

$$R_3 = U_3 - C_{32} = 1659.61 - 1600 = 59.61$$

$$R_4 = U_4 - C_{41} = 1363.41 - 1200 = 163.41$$

Problem 2 Life insurance - two state model

Let Y be the length of life for a representative individual in years (i.e. Y takes only values in natural numbers). For $l \in \mathbb{N}$ we define the probability of surviving at least one more year from age l as $p_l = \mathbb{P}(Y > l+1 | Y > l)$.

After discovery of a miraculous anti-aging medicine and making it available to the whole population, the statisticians estimated that this probability as a function of l is flat and not dependent on age l any more (i.e. $p_l = p$ for all $l \in \mathbb{N}$ for some $p \in (0, 1)$).

2a 2pt

Argue shortly why the flat function $l \mapsto p_l$ would not have been a realistic mortality model before discovery of this medicine? What relation between l and p_l would you expect before that discovery?

Solution: Without the medicine the probability of survival decreases to 0 with age as it is less and less likely that the individual would survive yet another year.

2b 3pt

For $k, l \in \mathbb{N}$ define ${}_k p_l$ as ${}_k p_l = \mathbb{P}(Y > l+k | Y > l)$. What is the interpretation of it? Give a general formula ${}_k p_l$ as a function of l, k and p and calculate the values for $l = 60$ and $k = 1, 2, 3, 4, 5$. Tip. You do not need to prove the formula for ${}_k p_l$ as long as you explain the idea behind it. Use $p = 0.98$.

Solution: ${}_k p_l = \mathbb{P}(Y > l+k | Y > l)$ is a probability of staying alive in next k years from year l . It can be computed using the formula that ${}_k p_l = (p_l) * \dots * (p_{l+k-1}) = p^k$. The idea behind formula is that surviving k next years from age l is equivalent to survive 1 year from age l , then surviving 1 year from age $l+1$, and so on and on up to surviving 1 year from age $l+k-1$. Probability of surviving 1 year from age $l+i$ is then equal to $p_{l+i} = p$.

Values of ${}_k p_{60}$ for $k = 1, 2, 3, 4, 5$:

$${}_1 p_{60} = p = 0.98$$

$${}_2 p_{60} = p^2 = 0.9604$$

$${}_3 p_{60} = p^3 = 0.9412$$

$${}_4 p_{60} = p^4 = 0.9224$$

$${}_5 p_{60} = p^5 = 0.9039$$

(Continued on page 7.)

2c 14pt

A defined benefit pension paid pays a fixed and recurring on a yearly basis amount B if a person reaches a certain retirement age l_r . The first payment is done on the day the insured reaches the retirement age l_r and recurs each year as long as the insured is alive.

Assume that a person who has just turned 30 has signed a defined benefit pension with the yearly benefit $B = 10000$ upon reaching the retirement age $l_r = 65$. The premium for this product is paid as a yearly premium paid in advance in each year as long as the insured is alive and has not reached the retirement age l_r . The interest rate is constant $r = 5\%$ and the probability of survival $p = 0.98$.

1. Explain the equivalence principle for calculating this premium. (2 pt)
2. Calculate this fair premium using the equivalence principle and the transition probabilities from point 2b. (6 pt)
3. Calculate the mathematical reserve for that policy? (6 pt)

Solution: The premium calculated according to the equivalence principle is a premium π such that the expected present value of all the cashflows is equal to 0. It means that the premium covers on average the expected benefits from the policy and in this sense it is a fair premium.

The expected present value of the cashflows in this case is as follows:

$$\begin{aligned} V_0 &= \sum_{k=0}^{l_r-l_0-1} \frac{-\pi * {}_k p_{l_0}}{(1+r)^k} + B * \sum_{k=l_r-l_0}^{\infty} {}_k p_{l_0} \frac{1}{(1+r)^k} \\ &= -\pi \sum_{k=0}^{l_r-l_0-1} \left(\frac{p}{(1+r)} \right)^k + B * \sum_{k=l_r-l_0}^{\infty} \left(\frac{p}{(1+r)} \right)^k \end{aligned}$$

Therefore the premium according to the equivalence principle ($V_0 = 0$) is equal to:

$$\pi = \frac{B * \sum_{k=l_r-l_0}^{\infty} \left(\frac{p}{(1+r)} \right)^k}{\sum_{k=0}^{l_r-l_0-1} \left(\frac{p}{(1+r)} \right)^k}.$$

Note that:

$$\sum_{k=l_r-l_0}^{\infty} \left(\frac{p}{(1+r)} \right)^k = \left(\frac{p}{1+r} \right)^{l_r-l_0} \frac{1}{1 - \frac{p}{1+r}}$$

and

$$\sum_{k=0}^{l_r-l_0-1} \left(\frac{p}{(1+r)} \right)^k = \frac{1 - \left(\frac{p}{1+r} \right)^{l_r-l_0}}{1 - \frac{p}{1+r}}$$

Therefore:

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$$\pi = B \frac{\left(\frac{p}{1+r}\right)^{l_r-l_0} \frac{1}{1-\frac{p}{1+r}}}{\frac{1-\left(\frac{p}{1+r}\right)^{l_r-l_0}}{1-\frac{p}{1+r}}} = B \frac{\left(\frac{p}{1+r}\right)^{l_r-l_0}}{1-\left(\frac{p}{1+r}\right)^{l_r-l_0}} = 981.65.$$

Mathematical reserve is equal to expected present value of future benefits, i.e. is equal to the one time premium paid in advance under equivalence principle. Therefore the mathematical reserve is equal to

$$\begin{aligned} \pi &= B * \sum_{k=l_r-l_0}^{\infty} {}_k p_{l_0} \frac{1}{(1+r)^k} = B * \sum_{k=l_r-l_0}^{\infty} \left(\frac{p}{1+r}\right)^k \\ &= B * \left(\frac{p}{1+r}\right)^{l_r-l_0} \frac{1}{1-\frac{p}{1+r}} = 13408.45 \end{aligned}$$

Problem 3 Life insurance - three state model

Consider now a three state insurance modelled for an active individual aged $l_0 = 60$. The Markov chain $(X_j)_{j=0,1,\dots}$ represents the state of the individual in j -years and can take three possible states: a - individual is alive and active, i - individual is alive and disabled, d - individual is dead.

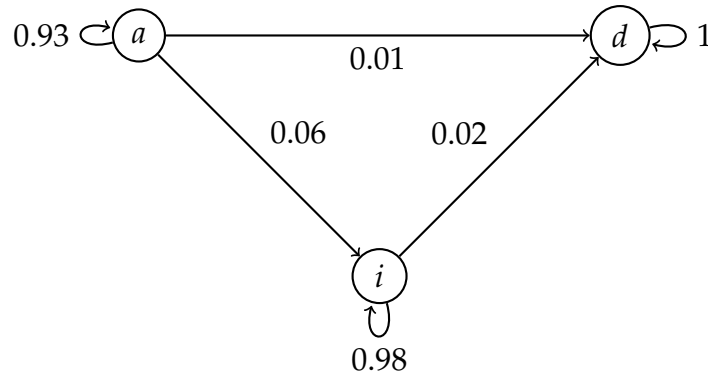
Let $p_{xy}(j) = P(X_j = y | X_{j-1} = x)$ for $x, y \in \{a, i, d\}$. Let $\mathbb{P}(j) = [p_{xy}(j)]_{x,y \in \{a,i,d\}}$ be a transition probability matrix for this Markov chain. Assume that for $j = 1, 2, 3, 4$ the matrix $\mathbb{P}(j)$ has a form:

$$\begin{aligned} \mathbb{P}(j) &= [p_{xy}(j)]_{x,y \in \{a,i,d\}} = \begin{matrix} & \begin{matrix} a & i & d \end{matrix} \\ \begin{matrix} a \\ i \\ d \end{matrix} & \begin{pmatrix} p_{aa}(j) & p_{ai}(j) & p_{ad}(j) \\ p_{ia}(j) & p_{ii}(j) & p_{id}(j) \\ p_{da}(j) & p_{di}(j) & p_{dd}(j) \end{pmatrix} \end{matrix} = \\ &= \begin{matrix} & \begin{matrix} a & i & d \end{matrix} \\ \begin{matrix} a \\ i \\ d \end{matrix} & \begin{pmatrix} 0.93 & 0.06 & 0.01 \\ 0.0 & 0.92 & 0.08 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \end{aligned}$$

3a 2pt

Draw a graph of this Markov chain and state which transitions are not possible.

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Solution: No transition from dead is possible. Moreover, the transition from inactive to active is not possible either.

3b 6pt

For $k = 1, 2, 3, 4$, calculate the probabilities $\mathbb{P}(X_k = i, X_{k-1} = a | X_0 = a)$, i.e. probability that a person is getting disabled at age $l_0 + k$ under the condition that right now that person is active and at age $l_0 = 60$.

Solution:

$$\mathbb{P}(X_k = i, X_{k-1} = a | X_0 = a) = (0.93)^{k-1} * 0.06$$

(We need to stay in "alive and active" for $k-1$ periods and then transition to "alive and inactive".)

Therefore

$$\mathbb{P}(X_1 = i, X_0 = a | X_0 = a) = 0.06$$

$$\mathbb{P}(X_2 = i, X_1 = a | X_0 = a) = (0.93) * 0.06 = 0.0558$$

$$\mathbb{P}(X_3 = i, X_2 = a | X_0 = a) = (0.93)^2 * 0.06 = 0.05189$$

$$\mathbb{P}(X_4 = i, X_3 = a | X_0 = a) = (0.93)^3 * 0.06 = 0.048261$$

3c 8pt

An active and alive individual aged $l_0 = 60$ has signed a disability insurance which is going to pay a one-time benefit 1000 if the insured has become disabled and did not reach the retirement age $l_r = 65$. The payment is made at the beginning of the period (e.g. if the insured was alive and active at 60 and is alive and disabled at age 61, he gets the payment when turning 61). The premium scheme for this policy is a one-time premium paid in advance.

1. Write the policy functions for this policy. (2pt)
2. Calculate the fair premium for this policy under the equivalence principle. Assume $r = 5\%$. (6pt)

(Continued on page 10.)

Solution: Policy functions:

$$a_a(k) = \begin{cases} -\pi & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_{ai}(k) = \begin{cases} 1000 & \text{if } k = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

$$a_i(k) = a_d(k) = a_{ad}(k) = a_{ia}(k) = a_{id}(k) = a_{da}(k) = a_{di}(k) = 0$$

Premium for this policy under equivalence principle is equal to

$$\pi = 1000 * \sum_{k=1}^4 \mathbb{P}(X_k = i, X_{k-1} = a | X_0 = a) \frac{1}{(1+r)^k} = 192.2879$$

Problem 4 Financial risk

Consider a derivative with maturity T written on the asset price S_T such the pay-off of the derivative is equal $g(S_T)$ for some function g . Assume that $S_T = s_0 \exp(\mu T + \sigma \sqrt{T} \epsilon)$, where μ and σ are parameters of the model and $\epsilon \sim N(0, 1)$ under a probability $\mathbb{P}$. Assume that r is a continuously compounded risk-free interest rate.

4a 3pt

Prove the following formula for the expected value of the lognormal distribution: if $X \sim LN(\mu, \sigma)$ then

$$\mathbb{E}(X) = \exp\left(\mu + \frac{\sigma^2}{2}\right).$$

Use it to calculate the expected value of S_T .

We remind that the density function for the normal distribution is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Solution: If $X \sim LN(\mu, \sigma)$ then $X = e^Y$ for $Y \sim N(\mu, \sigma)$

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}(e^Y) = \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{2x\sigma^2 - x^2 + 2x\mu - \mu^2}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-x^2 + 2x(\sigma^2 + \mu) - (\sigma^2 + \mu)^2 + (\sigma^2 + \mu)^2 - \mu^2}{2\sigma^2}\right) dx = \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x - (\sigma^2 + \mu))^2}{2\sigma^2}\right) dx \exp(\mu + \sigma^2/2) = \exp(\mu + \sigma^2/2) \end{aligned}$$

(Continued on page 11.)

4b 3pt

Under the risk-neutral probability Q , the discounted present value of S_T must be equal to s_0 in order for the model not introduce the arbitrage (i.e. profit without risk). Assume that S_T has the following form $S_T = s_0 \exp(\mu_Q T + \sigma \sqrt{T} \epsilon)$ under risk-neutral probability Q , where $\epsilon \sim N(0, 1)$ under a probability Q . What must be the value of μ_Q so that the abovementioned condition for arbitragefreeness is satisfied?

Solution: The discounted present value of S_T is equal to

$$\begin{aligned} V_0 &= e^{-rT} \mathbb{E}^Q[S_T] = e^{-rT} \mathbb{E}^Q[s_0 \exp(\mu_Q T + \sigma \sqrt{T} \epsilon)] \\ &= s_0 e^{-rT + \mu_Q T + \frac{\sigma^2 T}{2}} \end{aligned}$$

Therefore $V_0 = s_0 \Leftrightarrow \mu_Q = \left(r - \frac{\sigma^2}{2}\right)$.

That tells us that S_T under the probability Q has the following distribution $S_T = s_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma \sqrt{T} \epsilon\right)$ where $\epsilon \sim N(0, 1)$ under a probability Q

4c 3pt

Sketch an algorithm which would approximate the price of this derivative $\mathbb{E}^Q[g(S_T)]e^{-rT}$ via Monte Carlo simulation. **Solution:**

1. Inputs: σ, r, T, m, g
2. For j in 1 to m do
 - (a) Draw $\epsilon_j \sim N(0, 1)$,
 - (b) Set $X_j = g(s_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma \sqrt{T} \epsilon_j\right))$
3. Return $V_0^{MC} = e^{-rT} \frac{1}{m} \sum_{j=1}^m X_j = \frac{e^{-rT}}{m} \sum_{j=1}^m g\left(s_0 e^{\left(r - \frac{\sigma^2}{2}\right) T + \sigma \sqrt{T} \epsilon_j}\right)$

4d 10pt

Consider a call option with maturity T and strike K written on the asset price S_T with the pay-off $X_C = \max(S_T - K, 0)$. Alternatively, this call option can be expressed as option on return R with pay-off $X_C = \max(R - r_g, 0)s_0$, where r_g is the minimum return, R is a return over time-period T . The relation between strike and the minimum return is then $K = s_0(1 + r_g)$. Note that $R = S_T/s_0 - 1$.

Present the proof of the Black-Scholes formula for V_0^C , i.e. the value of the call option at time $t = 0$ (i.e. derive it from scratch).

Solution:

(Continued on page 12.)

Let $\phi(x)$ be a standard normal density and $\Phi(x)$ be a cumulative distribution function of a standard normal distribution. We compute:

$$\begin{aligned} V_0 &= e^{-rT} \mathbb{E}^Q[\max(S_T - K, 0)] \\ &= e^{-rT} \mathbb{E}^Q \left[\max \left(s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} \epsilon \right) - K, 0 \right) \right] \\ &= e^{-rT} \int_{-\infty}^{\infty} \left[\max \left(s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) - K, 0 \right) \phi(x) \right] dx \end{aligned}$$

Note that if x satisfies following $s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) - K < 0$ then the expression under the integral is 0. Transforming this we get the following condition for integral to be 0:

$$x < a = \frac{\ln \left(\frac{K}{s_0} \right) - \left(r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

Returning to our integral we get

$$\begin{aligned} V_0 &= e^{-rT} \int_{-\infty}^{\infty} \left[\max \left(s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) - K, 0 \right) \right] \phi(x) dx \\ &= e^{-rT} \int_a^{\infty} \left[s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) - K \right] \phi(x) dx \\ &= e^{-rT} \int_a^{\infty} \left[s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) \right] \phi(x) dx \\ &\quad - e^{-rT} K \int_a^{\infty} \phi(x) dx \\ &=: I_1 - I_2 \end{aligned}$$

Focus first on I_2 . Using a relation $1 - \Phi(x) = \Phi(-x)$ we get:

$$I_2 = e^{-rT} K \int_a^{\infty} \phi(x) dx = K e^{-rT} (1 - \Phi(a)) = K e^{-rT} (\Phi(-a))$$

We also have

(Continued on page 13.)

$$\begin{aligned}
I_1 &= e^{-rT} \int_a^\infty \left[s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) \right] \phi(x) dx \\
&= e^{-rT} \int_a^\infty \left[s_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} x \right) \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= s_0 e^{-rT + \left(r - \frac{\sigma^2}{2} \right) T} \int_a^\infty e^{\sigma \sqrt{T} x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= s_0 e^{-\left(\frac{\sigma^2}{2} \right) T} \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 - 2\sigma \sqrt{T} x}{2}} dx \\
&= s_0 e^{-\frac{\sigma^2 T}{2}} \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 - 2\sigma \sqrt{T} x + \sigma^2 T - \sigma^2 T}{2}} dx \\
&= s_0 e^{-\frac{\sigma^2 T}{2}} e^{-\frac{\sigma^2 T}{2}} \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \sigma \sqrt{T})^2}{2}} dx \\
&= s_0 \int_{a - \sigma \sqrt{T}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = s_0 (1 - \Phi(a - \sigma \sqrt{T})) = s_0 (\Phi(-a + \sigma \sqrt{T}))
\end{aligned}$$

Summarizing we get

$$V_0 = s_0 \Phi(-a + \sigma \sqrt{T}) - K e^{-rT} \Phi(-a)$$

4e 3pt

What is a call-put parity? Use it to calculate the price of put option given the price of a call option.

Solution: A put-call parity is the relation between a price put V_0^P and call option V_0^C on the same underlying instrument S , with the same maturity T and the same strike K (alternatively the same guaranteed return r_g):

$$V_0^C = V_0^P + s_0 - K e^{-rT}$$

Therefore by rearranging and plugging in B-S from point d):

$$\begin{aligned}
V_0^P &= V_0^C - s_0 + K e^{-rT} = s_0 \Phi(-a + \sigma \sqrt{T}) - K e^{-rT} \Phi(-a) - s_0 + K e^{-rT} \\
&= -s_0 (1 - \Phi(-a + \sigma \sqrt{T})) + K e^{-rT} (1 - \Phi(-a)) = \\
&= -s_0 \Phi(a - \sigma \sqrt{T}) + K e^{-rT} \Phi(a)
\end{aligned}$$

The last equality follows from $1 - \Phi(x) = \Phi(-x)$.

Proof of Call-put parity (optional, do not subtract if parity is not proved):

Introduce notation $X_P = \max(K - S_T, 0)$, $X_C = \max(S_T - K, 0)$. Note that $X_C - X_P = S_T - K$ Using non-arbitrage condition in 4b one gets:

$$\mathbb{E}^Q[e^{-rT}(X_C - X_P)] = \mathbb{E}^Q[e^{-rT}(S_T - K)] = s_0 - K e^{-rT}$$

(Continued on page 14.)

On the other hand

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT}(X_C - X_P)] = \mathbb{E}^{\mathbb{Q}}[e^{-rT}X_C] - \mathbb{E}^{\mathbb{Q}}[e^{-rT}(X_P)] = V_0^C - V_0^P$$

Comparing both equations one gets put-call parity

END