

Introduction to insurance mathematics

Autumn 2024, UiO

Formalities

- **Classes:** Mondays 14:15-15:50 and Tuesday 8:25-10:00 (5 min break)
- **Textbook:** Erik Bølviken "Computation and Modelling in Insurance and Finance", Cambridge University Press, 2014
- **Exam:** November 29th at 15:00 (4h). **Location:** Silurveien 2 Sal 4A
- **Mandatory assignment (most likely):** October 24th, 14:30 (will come back to it)
- **Website:**
<https://www.uio.no/studier/emner/matnat/math/STK3505/h24/index.html>
- Presentations, solutions of exercises and other info will be published on the website regularly

Formalities, continued

- The course will cover the material from the textbook, but will be presented in slightly different order and manner. Information on the material covered in each week on the website.
- The lectures will extend/supplement the textbook with some basic notions about insurance and info about real life problems. Some of it will be required on the exam.
- Material required during the exam will be marked in the final presentation with the red rectangular in the corner.

Actuarial science

What it is?



Actuarial science is the discipline that applies **mathematical and statistical methods** to assess risk in insurance, pension, finance, investment and other industries and professions. More generally, actuaries apply rigorous mathematics to model matters of **uncertainty**.

https://en.wikipedia.org/wiki/Actuarial_science



Actuarial science includes a number of interrelated subjects, including **mathematics, probability theory, statistics, finance, economics, financial accounting and computer science.**

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Historically, actuarial science used **deterministic models** in the construction of tables and premiums. The science has gone through revolutionary changes since the 1980s due to the proliferation of high speed computers and the union of **stochastic actuarial models** with **modern financial theory**.

https://en.wikipedia.org/wiki/Actuarial_science



What is differentiating *actuarial science* from *mathematical finance* is the focus on **risk sharing and diversification** in order to mitigate the adverse effect of events of uncertain occurrence, severity and timing.

Structure of the course



1. Introduction
2. Finance, Part 1:
 - ⊙ Time value of money
 - ⊙ Monte Carlo,
 - ⊙ Distributions and basic models for assets
3. Non-Life insurance:
 - ⊙ Compound Poisson model and Monte Carlo
 - ⊙ Premium calculation
 - ⊙ Reserving
 - ⊙ Reinsurance
 - ⊙ How it works in practice
4. Life insurance
 - ⊙ Life-tables, discounting under mortality
 - ⊙ Reserves and premium calculation
 - ⊙ Markov chains in life-insurance
 - ⊙ Monte Carlo simulation
 - ⊙ Modern insurance products
5. Finance, Part 2:
 - ⊙ Options and Black-Scholes formula
 - ⊙ Price of guaranteed returns in life insurance products via options

Insurance

What is it and how does it work?



Book: Chapter 1, sections 1.1–1.2



Basic idea behind insurance –

sharing the risk across many individuals in order to be able to cover unlikely but substantial losses



Works because of the Law of Large Numbers



Basic mathematic model

- ① $X_1, X_2, \dots, X_n$ - losses from policy 1, 2, ..., n (assumed to be i.i.d.)
- ② $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i = E(X_1) =: \pi^{pu}$ **pure premium**
- ③ Insurance company in this model does not lose (or profit) almost surely if it charges pure premium and has infinite amounts of customers
- ④ In practice:
 - ⑤ no portfolio is infinite
 - ⑥ losses can occur at the same time (e.g. NatCats, Covid-19),
 - ⑦ different policies with different risk (but this is less of a problem by proper risk differentiation)
 - ⑧ Insurance companies want to have some profit
- ⑨ Therefore: premium paid has a loading γ such that $\pi = \pi^{pu}(1 + \gamma)$

Insurance

Non-life insurance/ Property & Casualty (P&C)

Usually **1-year** cover against unlikely risks with adverse financial effect

- Car insurance (3rd party liability, casco)
- Property (building, contents)
- Travel insurance
- Workers Compensation (yrkesskade)
- Health-related products
- Specialistic covers (liability, pet, marine, agriculture, business interruption)
- (Collective) death insurance

Life insurance

Usually **longer term contracts** which protect against events where the uncertainty is both **whether** and **when** the trigger event occurs

- Death insurance/Term-insurance
- Pure endowment insurance
- Endowment insurance
- Pension – “insurance against living for too long time” ;-)
- Disability insurance
- Nowadays also: saving and investment without any particular “insurance” element

Role of an actuary



Pricing actuary

- Building models which predict **risk level given risk factors** (e.g. age of a driver)
- Determining premium level for different groups (**tariffs**) based on those models
- Risk selection (what to insure, what not to)
- Follow up of the exposure to detect changes and update models

Reserving actuary

- Building models to set up **reserves**
- Reserves cover **expected future payments** for claims which have not happened, or which have happened but the amount to be paid is uncertain
- Compare the estimates from pricing against experience

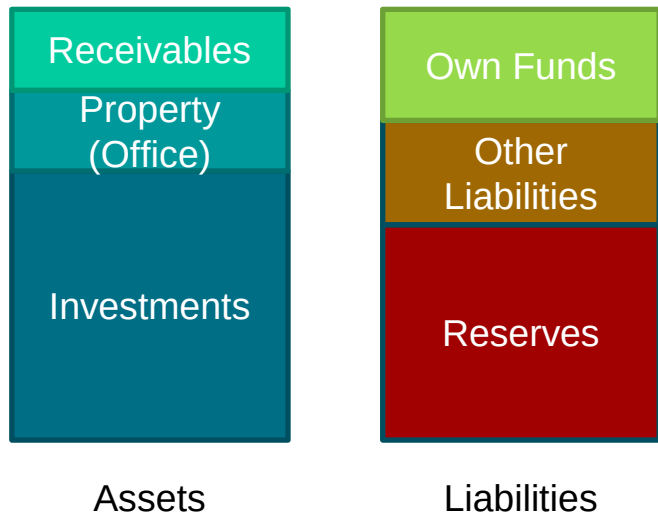
Capital actuary

- Building models to determine how much capital to hold to cover **unexpected losses**, i.e. future payments which exceeds reserves
- Advising about the risk appetite (i.e. how much risk to take given available capital) and how to use capital efficiently

Accounting in the actuarial science

Why?

Balance sheet of an insurance company



Assets: what a company "owns"

- Investments, property, cash
- "Receivables", i.e. what others owe to an insurance company, e.g. not paid premiums

Liabilities: what a company owes to others

- Reserves** (AKA "**technical provisions**")
- Other liabilities (e.g. loans or debt)

Equity or Own Funds: net wealth

- Simplistically: assets minus liabilities
- Can be seen as what a company owes to its owners

Change in equity = profit = investment return + profit on UW +...

Why an actuary needs to know a bit of accounting?



Pricing actuary in Non-Life:

- ☉ not really that important 😊

Reserving and capital actuaries, pricing actuaries in Life:

- ☉ reserves are one of the main elements on the liability side and determine level of equity and profit
- ☉ Different standards and rules
 - ☉ IFRS (International Financial Reporting Standard)
 - ☉ Local GAAP (Generally Accepted Accounting Principles)
 - ☉ Solvency or tax purposes
- ☉ For Life-insurance: interaction between profits, reserves and benefits.

Finance and the actuarial science

What's the connection here?

Why finance?



- Reserves can be seen as **cashflows** (i.e. a sequence of future payments/money transfers happening at different times). Therefore natural to treat them in financial terms and e.g. to discount them.
- Investments – appropriate to the nature of the reserves and contracts
 - More about that during the first exercise classes
- Interaction between reserves and investment returns, especially for life insurance
 - Cashflows may depend on the result from investments. Then the amount of the cashflow can be seen as a financial derivative/contingent claim

Life-insurance products as saving / investment



Term-insurance

- Pays an agreed amount (benefit) if insured dies in an agreed period
- Insurance against dying
- Uncertain if benefit is paid and when

Pure endowment

- Pays a benefit if insured is alive at the end of an agreed period
- Insurance against surviving
- Uncertain if benefit is paid, no uncertainty about timing

Endowment

- Combination of term and pure endowment
- Pays a benefit regardless if insured dies or survives it
- No uncertainty regarding the occurrence of benefit, but only about timing

Endowment insurance as a form of saving/ investment



Assume single premium $\pi = 1000$ paid at the beginning of the contract and benefit $B=1300$, paid out if an insured dies before next 15 years or after 15 years if an insured survives.

If the insured survives, then she will yield an annually compounded return

$$\sqrt[15]{B/\pi} - 1 = 1,76\%$$

If the insured dies after 5 years, then her heirs will yield an annually compounded return equal to

$$\sqrt[5]{B/\pi} - 1 = 5,39\%$$

Risk sharing here is between those who survive and get possibly lower returns, and those who die and whose heirs get higher returns.

Time value of money

I.e. why \$1000 today is not the same as \$1000 in a year



Book, Chapter 1, sections 1.3,
1.4 (without 1.4.3)

Concept



- ① An amount of money today is worth more* now than the same amount will be worth in the future
- ② This is due to the fact that we can invest the money today and earn potential profit on this investment
- ③ To compare to amounts of money at two different time points we need to take that profit into account

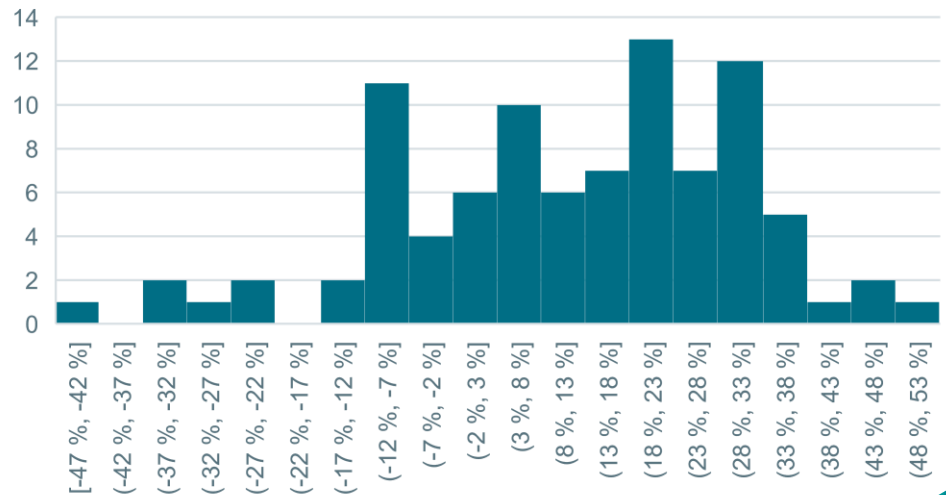
* We assume here that interest rates are non-negative ;-)

Risk-free interest rate vs. risky assets



- Most of the assets are risky, i.e. we don't really know their future value nor return
- If we invest \$1000 e.g. in stocks today equally plausible that in a year we have \$900 or \$1100. Difficult to use risky assets to assess time-value of money.
- We need to use an asset with a "risk-free" return, e.g. sovereign bonds or "bank account".

Histogram of S&P500 yearly returns from 1929





Principal, interest and compounding

- ① **Principal** or capital is an amount of money which the interest is calculated upon. E.g. I open a saving account in a bank and put 1000 kr on it. The principal then is 1000 kr.
- ② **Interest** is an amount of money which you earn on principal by holding it on that saving account. E.g. That saving account offers 12% yearly interest, so after a year I will earn $12\% \cdot 1000 = 120$ kr.
- ③ **Compounding** interest (or capitalization) is time after which the earned interest is added to the principal. E.g. if 12% is earned with yearly compounding, after a year the principal will grow to 1120 kr.

On interest rate and return – annual compounding

- ① Assume we put \$1000 on a bank account with interest rate 12% and yearly capitalization.
- ② After a year we will have therefore \$1120. If the interest rate is unchanged and we keep all the money and the proceeds on the account, after another year we will get \$1254,4, i.e. 12% of \$1120.

Annual compounded interest rate:

$$V_K = V_0(1 + r)^K$$

V_t - Value of the bank account at time t (in years), $t=0,1,2,3,\dots$

r – interest rate (with annual capitalization of interest)

Present value of a cashflow – discounting with the annually compounded interest



- ① We know that the amount X at time 0 is worth $X(1 + r)^K$ after K years
- ① How much is the amount Y which we can get in K years, worth now?

$$\frac{Y}{(1 + r)^K}$$

- ① This is known as a **present value (PV)** of a future cashflow or **discounting** of a future cashflow

On interest rate and return – monthly compounding

- Assume we put \$1000 on a bank account with yearly interest rate 12% and monthly capitalization.
- After a month we will have therefore \$1010, i.e. 12% / 12. If the interest rate is unchanged and we keep all the money and the proceeds on the account, after another month we will get \$1020,1, i.e. 1% of \$1010.
- After two years (with interest rate unchanged) we would get \$1269,735

Monthly compounded interest rate:

$$V_K = V_0 \left(1 + \frac{r}{12}\right)^{K*12}$$

V_t - Value of the bank account at

time t (in years), $t = 0, \frac{1}{12}, \frac{2}{12}, \dots$

r – annual interest rate (with monthly capitalization of interest)

On interest rate and return – daily and continuous compounding



Similarly, in case of daily capitalization the value of the bank account can be calculated as

$V_K = V_0(1 + r * p)^{K/p}$, where
 $p = 1/365$ is the fraction of the year after which the interest is capitalized

What would happen if interest is capitalized instantaneously? That would mean $p \rightarrow 0$

$$\lim_{p \rightarrow 0} V_0(1 + r * p)^{K/p} = V_0 e^{rK} = V_K$$

Therefore \$1000 after 2 years with 12% continuously compounded interest is:

$$1000e^{2*12\%} = \$ 1271,25$$

Note also that we can measure the value at any non-negative time K.

Annual/continuous compounding vs. arbitrage



Arbitrage = profit without risk

Assume annually compounded and continuously compounded interest rates are both 12%. Then I could borrow \$1000 with annual compounding for two years and put it on the bank account with continuous compounding for two years.

After two years I need to return \$1254,4, but I have \$1271,25 on my bank account. I.e. without taking any risk I gained \$16,85, i.e. managed to find arbitrage opportunity.



Annual/continuous compounding vs. arbitrage

Markets with arbitrage shouldn't exist, and therefore annually compounded interest rate must be higher than continuously compounded interest rate.

Assume that a annually compounded interest rate r_a is 12%. What is the continuously compounded interest rate r_c which leads to arbitrage-free market?

We know that that after K years with annually compounded interest we have $V_0(1 + r_a)^K$ whereas with continuous compounding we get $V_0 e^{r_c K}$. To avoid arbitrage both amounts must equal. Easy to check that: $r_c = \ln(1 + r_a) = 11,33\%$.

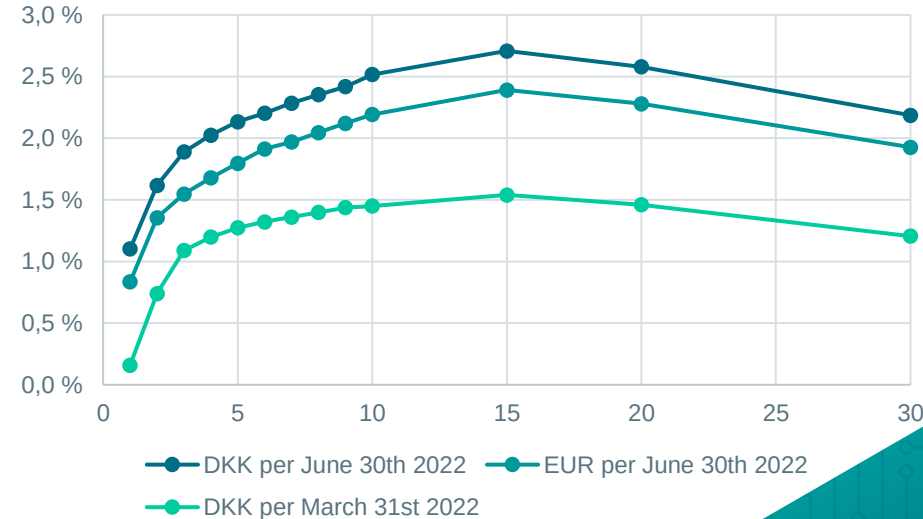
Note that we can also discount with continuous compounding, but we need to use the proper interest rate!!!!

Is r the same regardless timing of cashflow? – zero coupon bonds



- Rate on the bank account can change ("refinancing risk") so no reason to assume that
- Zero-coupon bond** – an instrument which pays you at maturity K an amount 1 (and nothing earlier or later). We assume no risk, i.e. we get 1 with probability 1
- $P_0(0:K)$ - price (at time 0) of zero-coupon bond with maturity K , we have it from the market
- $P_0(0:K) = 1 * (1 + \bar{r}_0(0:K))^{-K}$
- So: $\bar{r}_0(0:K) = (P_0(0:K))^{-1/K} - 1$
- $\bar{r}_0(0:K)$ not necessarily equal $\bar{r}_0(0:L)$, $K \neq L$
- Interest rate curve:** $t \mapsto \bar{r}_0(0:t)$
- Discounting with market interest rates leads to **fair valuation** or **market value** of a cashflow

Interest rate curves for DKK and EUR



Bonds and duration



Bond (in general) is an instrument paying a fixed stream of payments $B_1, B_2, \dots, B_K$ at some predefined times $t_1, t_2, \dots, t_K$.

● Bond = “portfolio of zero-coupon bonds”

● **Value of the bond:**

$$V_0 = \sum_{i=1}^K B_i * P_0(0: t_i)$$

● **Yield** of the bond: the solution (in y) of

$$V_0 = \sum_{i=1}^K \frac{B_i}{(1+y)^{t_i}}$$

If interest rate curve flat yield y = interest rate r .

This is what we mostly assume in this course!!!!

Duration of a bond (or sequence of cashflows): average time when payments occur, weighted with the discounted size of the payment

$$D = \sum_{i=1}^K t_i * q_i, \text{ where } q_i = \frac{B_i(1+y)^{-t_i}}{\sum_{i=1}^K B_i(1+y)^{-t_i}} = \frac{B_i(1+y)^{-t_i}}{V_0}.$$

Note that the price changes if interest rate/yield changes. What is the sensitivity of the price to the yield?

Easy to prove that:

$$\frac{d}{dy} V_0 = \frac{-D}{1+y} V_0.$$

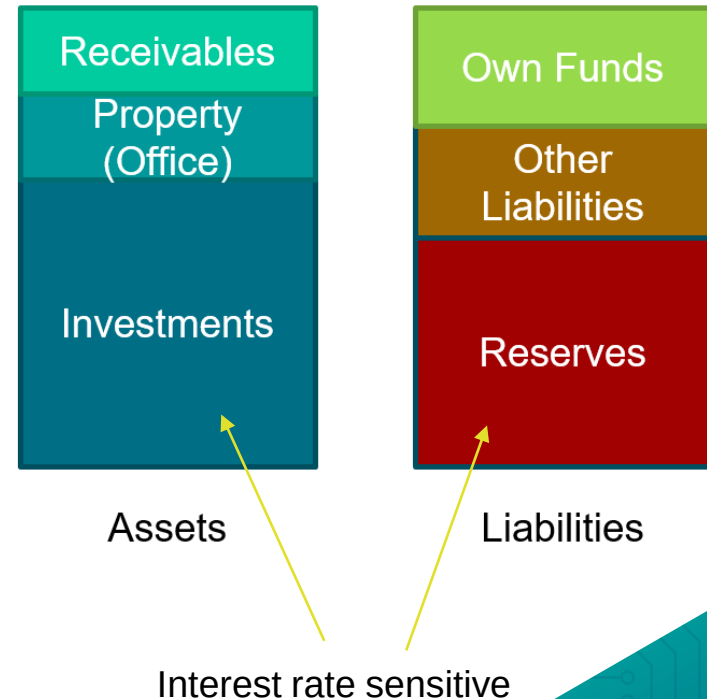
Therefore: $\Delta V_0 \approx \frac{-D}{1+y} V_0 * \Delta y$.

Interpretation: Take a bond with duration 5 years and price $V_0 = 100$. Assume $r=y=3\%$. If interest rate increases with 1 **percentage point** (i.e. to 4%), the price of the bond will decrease by approximately

$\frac{D}{1+y}$ **percent** (i.e. $5/(1,03) \%$). New price is $100 * (1 - 5/1,03 * 0,01) = 95,15$.

Insurance reserves as cashflows

- We mentioned that reserves for the future benefits or claims can be seen as a sequence of cashflows.
- If we assume that this sequence of cashflows is deterministic, we can treat it just like a bond and discount, compute duration, interest rate sensitivity.
- Using duration, we can therefore measure how much our assets and liabilities change by change of interest rate and therefore how much our Own Funds (=Assets – Liabilities) vary



Risky assets, portfolios and returns

Risky assets



- So far we mostly considered risk-free “fixed” interest rate and zero-coupon bonds
- Most of assets carry risk regarding future value:
 - Risky bonds (e.g. corporate bonds)
 - Stocks
 - Property
 - Commodities
- Even the future value of a zero-coupon bond is not constant/deterministic before the maturity. Only at maturity we know for sure the value of a zero-coupon bond (i.e. 1).
- Let V_0 be a value of an asset at time 0. Let V_1 be a value of the same asset at time 1.
- The return R of this asset is defined as relative change in the value, i.e. $R = \frac{V_1 - V_0}{V_0}$.
- At time 0 we don't know V_1 so it is a random variable, and therefore R is a random variable too.
- At time 1, we know the realisation of $V_1(\omega)$. Therefore we know therefore also realisation of $R(\omega)$.

Portfolios and multi-period returns



We can have a **portfolio** (i.e. basket) of different assets. Let V_0^j be a value invested in an asset j at time 0.

Then $V_0 = \sum_{j=1}^J V_0^j$ and $w_j = \frac{V_0^j}{V_0}$ is the weight of asset j in the portfolio (at time 0)

Let $V_1 = \sum_{j=1}^J V_1^j$ be the value at time 1.

Easy to prove that $V_1 = \sum_{j=1}^J (1 + R^j) w_j V_0$, where R^j is a return for asset j .

In other words $R = \sum_{j=1}^J R^j w_j$

Similarly we can consider multi-period returns.

Let V_k value of the investment in an asset at time k .
Let R_k a return which that assets earns between time $(k-1)$ and k .

Then $V_K = V_0(1 + R_1)(1 + R_2) \dots (1 + R_K)$.

Overall return between 0 and K (K -step return in the book) can be expressed as:

$$R_{0:K} = (1 + R_1)(1 + R_2) \dots (1 + R_K) - 1$$

If the step is a year, we can be interested in average annual return $\bar{R}_{0:K}$. Then

$$\bar{R}_{0:K} = [(1 + R_1)(1 + R_2) \dots (1 + R_K)]^{\frac{1}{K}} - 1 \text{ and} \\ (1 + \bar{R}_{0:K})^K = (1 + R_{0:K}).$$

Lets simplify it a bit with log-returns



Let $X_k = \ln(1 + R_k)$ – a **log-return** in period between $k-1$ and k .

Let $X_{0:K} = \ln(1 + R_{0:K})$

Easy to check that $X_{0:K} = X_1 + \dots + X_K$

Similarly, Let $\bar{X}_{0:K}$ be the average annual log-return in period between 0 and K, defined as

$\bar{X}_{0:K} = \ln(1 + \bar{R}_{0:K})$.

Then

$$\begin{aligned}\bar{X}_{0:K} &= \ln(1 + \bar{R}_{0:K}) = \ln\left([(1 + R_1)(1 + R_2) \dots (1 + R_K)]^{\frac{1}{K}}\right) = \frac{1}{K} [\ln(1 + R_1) + \dots + \ln(1 + R_K)] = \\ &= \frac{1}{K} [X_1 + \dots + X_K]\end{aligned}$$