

Monte Carlo simulation



Book, chapter 2, sections 2.1-2.2
and 2.3.2



Monte Carlo simulation

Monte Carlo method is an algorithm based on repeated sampling/simulating a “system” in order to investigate the resulting distribution of results.

The idea:

1. We want to investigate a random vector X , however we don't know its distribution
2. We know however there is a random vector Y and a function G such that Y is easy to sample from and $X=G(Y)$.
3. So let's simulate Y by sampling $Y_1^*, Y_2^*, \dots, Y_m^*$, calculate $X_i^* = G(Y_i^*)$ and investigate the properties of $X_1^*, X_2^*, \dots, X_m^*$



Monte Carlo mean and standard dev.

MC mean

Estimator $\bar{X}^* = \frac{1}{m} \sum_{i=1}^m X_i^*$ is an unbiased estimator, i.e.

$E[\bar{X}^*] = \mu$, where $\mu = E[X]$.

By LLN the estimator is also consistent (i.e. converges to the true value as m goes to infinity).

Easy to check that

$sd[\bar{X}^*] = \frac{\sigma}{\sqrt{m}}$, where $\sigma = sd(X)$.

MC standard deviation

Estimator $s^* = \sqrt{\frac{1}{m-1} \sum_{i=1}^m (X_i^* - \bar{X}^*)^2}$ is

an asymptotically unbiased estimator of true standard deviation σ .

It is also consistent and approximately

$sd[s^*] \approx \frac{\sigma}{\sqrt{2m}} \sqrt{1 + \kappa/2}$, where $\kappa = \text{excess kurtosis}(X)$.

About estimation error



It can be much easier to estimate correctly second order estimates (e.g. st.deviation) than first-order parameters (e.g. mean), especially if true mean is much lower than the true standard deviation

$$\frac{sd[\bar{X}^*]}{\mu} = \frac{\sigma}{\mu} \frac{1}{\sqrt{m}}$$

$$\frac{sd[s^*]}{\sigma} \approx \frac{1}{\sqrt{m}} \sqrt{\frac{1}{2} + \kappa/4}$$



Building confidence intervals - mean

Both MC mean and standard deviation approximately normal as $m \rightarrow \infty$.

Then we know that $(1 - \alpha)$ -confidence interval can for MC mean be defined as

$\bar{X}^* - U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{m}} < \mu < \bar{X}^* + U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{m}}$, where $U_{\frac{\alpha}{2}}$ is $(1 - \frac{\alpha}{2})$ - *quantile* of a standard normal distribution,

This is called $(1 - \alpha)$ -confidence interval as

$$P\left(\bar{X}^* - U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{m}} < \mu < \bar{X}^* + U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{m}}\right) \approx 1 - \alpha.$$

Note that $U_{2,5\%} \approx 1,96 \dots \approx 2$, therefore formula from the book for 95%-confidence interval is the same as here.

Similarly standard deviation



Then we know that $(1 - \alpha)$ -confidence interval for MC standard deviation can be defined as $s^* - U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{2m}} \sqrt{1 + \kappa/2} < \sigma < s^* + U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{2m}} \sqrt{1 + \kappa/2}$, where $U_{\frac{\alpha}{2}}$ is $(1 - \frac{\alpha}{2})$ - *quantile* of a standard normal distribution,

Similarly to mean

$$P\left(s^* - U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{2m}} \sqrt{1 + \kappa/2} < \sigma < s^* + U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{2m}} \sqrt{1 + \kappa/2}\right) \approx 1 - \alpha.$$

Note that $U_{2,5\%} \approx 1,96 \dots \approx 2$, therefore formula from the book for 95%-confidence interval is the same as here.



Percentile/quantile

α – **quantile** of the distribution with CDF F is defined as an amount q_α such that

$$F(q_\alpha) = \alpha$$

Percentile is formally the same but α is expressed as percentage, i.e. 5-percentile is 5%-quantile. The book introduces lower and upper percentiles/quantiles, but we will just focus on lower quantiles.

Quantiles measure the tail of the distribution, i.e. what is the worst loss we can expect with the given probability.

α – quantile in financial world is often called often Value-at-Risk (VaR) at a given confidence level.

Solvency Capital Requirement is based on VaR of a change of Own Funds (i.e. loss of capital) on 1-yr basis on confidence level 0,5% (i.e. 1-in-200 year loss).

MC quantile



Assume we have MC simulations $X_1^*, X_2^*, \dots, X_m^*$. We would like to estimate α –quantile from that simulation set.

The algorithm:

1. First calculate so-called **order statistics** $X_{(1)}^*, X_{(2)}^*, \dots, X_{(m)}^*$, i.e. find such a sequence $i_1, i_2, \dots, i_m$ such that
 - ⊙ $X_{(i_k)}^* = X_k^*$ and
 - ⊙ $X_{(1)}^* \geq X_{(2)}^* \geq \dots \geq X_{(m)}^*$
2. We define MC quantile q_α^* as

$$q_\alpha^* = X_{(m(1-\alpha))}^*$$

MC properties



q_{α}^* is asymptotically unbiased and consistent, i.e. expected value of q_{α}^* converges to the true value as $m \rightarrow \infty$ as well as q_{α}^* converges to the true value almost surely

Error:

$$sd(q_{\alpha}^*) \approx \frac{\sqrt{\alpha(1-\alpha)}}{f(q_{\alpha})} \frac{1}{\sqrt{m}},$$

Where f is a density function of X . This is unknown but we can have some estimates.



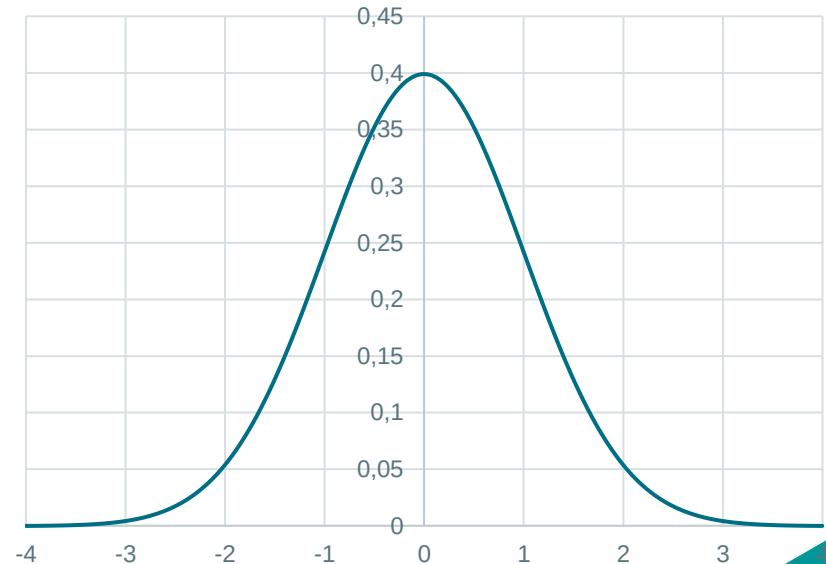
Density estimate with Gaussian kernel

Gaussian density function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Idea: try to estimate the density of X from the MC simulation set by weighting appropriately Gaussian densities

Gaussian probability density function



MC density estimate



$$f^*(x) = \frac{1}{m} \sum_{i=1}^m \frac{1}{hs^*} \phi\left(\frac{x - X_i^*}{hs^*}\right), \text{ where } h \text{ is a smoothing parameter.}$$

$f^*(x)$ is a biased estimator with

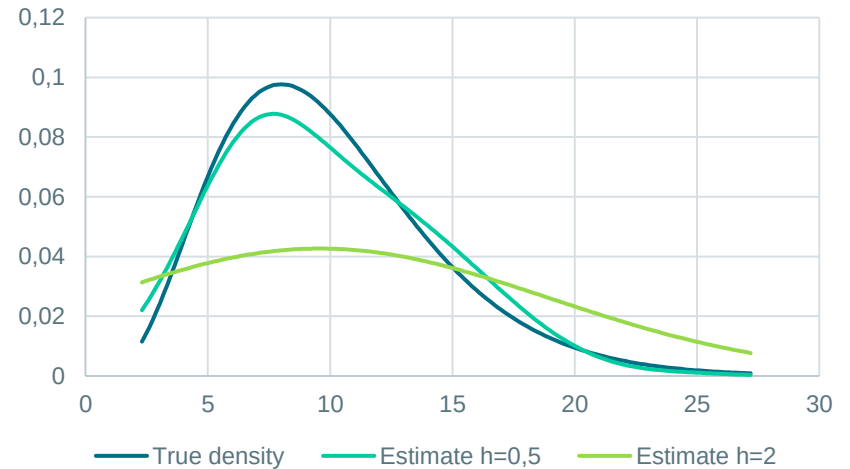
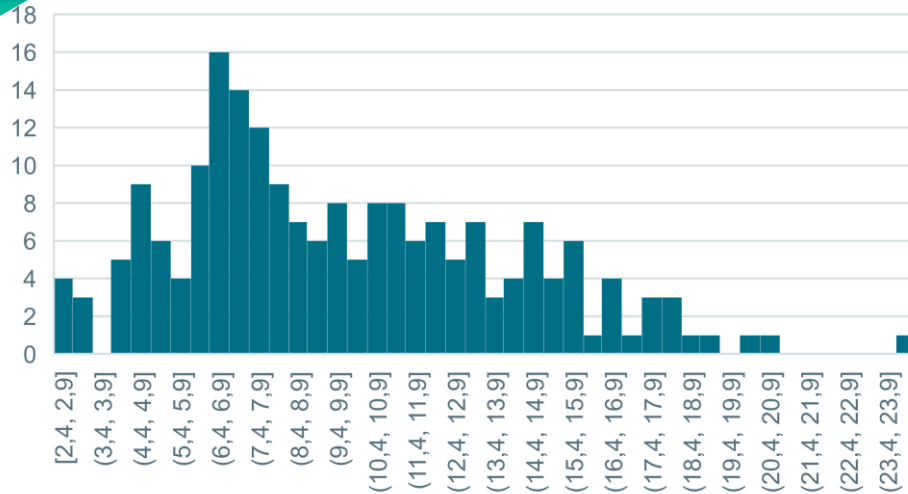
$$E(f^*(x)) \approx f(x) + \frac{1}{2} h^2 \sigma^2 \frac{d^2 f(x)}{dx^2}$$

And

$$sd(f^*(x)) \approx \frac{0.5311}{\sigma} \sqrt{\frac{f(x)}{hm}}$$

Example, $S^*=4,2$

Histogram



Sampling using the inverse method



Given a distribution F how to draw random number from that distribution?

It can be proved that the strictly increasing cumulative distribution function F has an inverse $F^{-1}(x)$ such that $F(F^{-1}(x)) = x$.

First note that $F(X) \sim U$ as

$$P(F(X) \leq u) = P(X \leq F^{-1}(u)) = F(F^{-1}(u)) = u \text{ if } u \in (0,1)$$

Therefore $X \sim F^{-1}(U)$.



Sampling algorithm for inverse method

1. Input: The percentile function ($F^{-1}(x)$)
2. Draw $U^* \sim \text{Uniform}$
3. Return $X^* \sim F^{-1}(U^*)$ or $X^* \sim F^{-1}(1 - U^*)$

MC and finance



Book: Chapter 2, section 2.4

Gaussian/normal distribution



Density function for normal distribution with mean μ and standard deviation σ

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

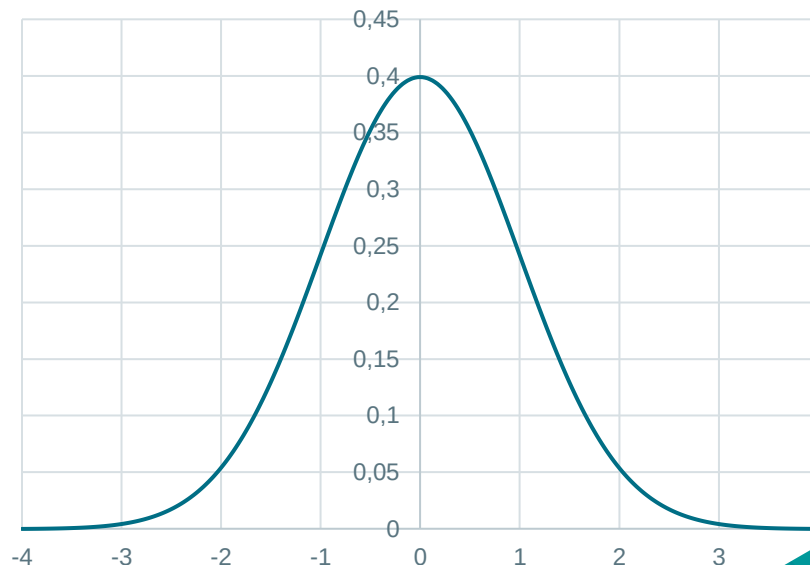
Closed form for CDF does not exist, however good algorithms to generate both CDF and its inverse.

Therefore possible to generate random variables via inverse method.

In R:

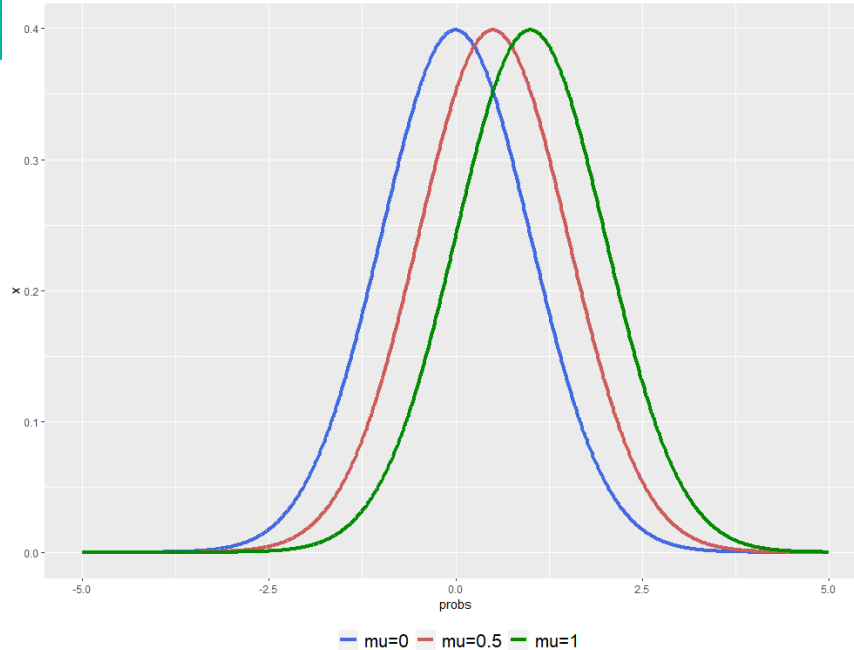
`pnorm`, `dnorm`, `qnorm`, `rnorm`

Standard normal density function

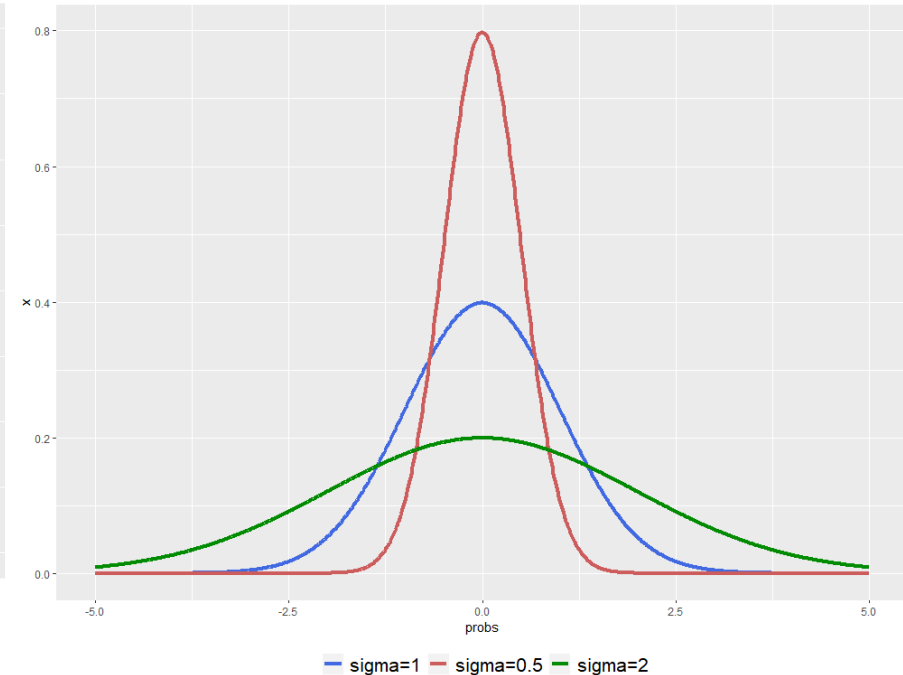


Effect of the parameter changes

Normal density for different μ , $\sigma=1$



Normal density for $\mu=0$ and different σ





Log-normal distribution

- Very common distribution such that $X = \exp(Y)$, $Y \sim N(\mu, \sigma)$. We say then that X is log-normally distributed with parameters μ and σ .

- $E(X) = e^{\mu + \frac{\sigma^2}{2}}$

- $sd(X) = E(X) \sqrt{e^{\sigma^2} - 1}$

- Distribution has a positive skew and fat right tail

- Drawing easy via normal distribution

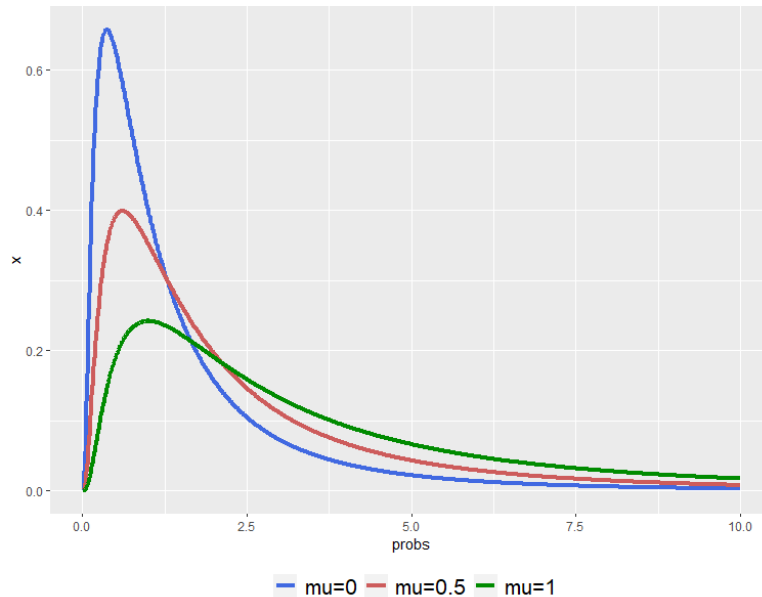
- Density

$$\phi(x) = \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}, x > 0.$$

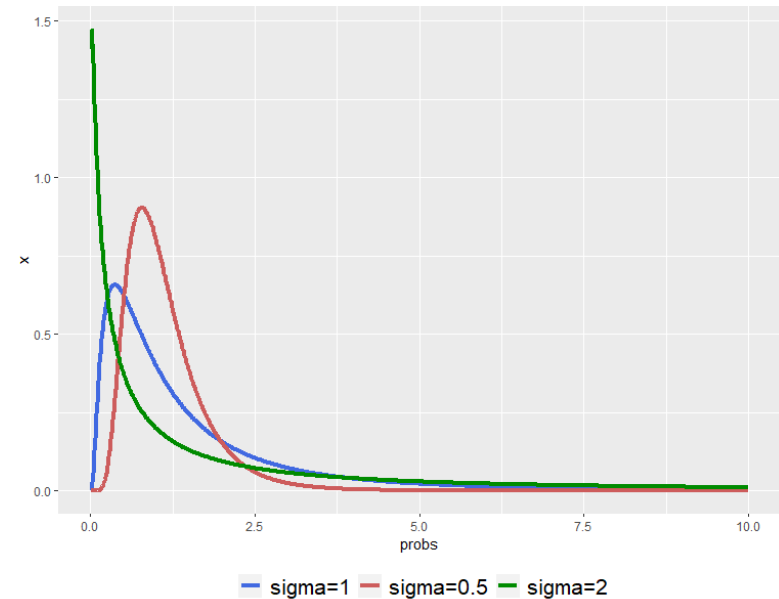
- In R: dlnorm, plnorm, qlnorm, rlnorm

Effect of the parameter change

Log-normal density for different μ , $\sigma=1$



Log-normal density for $\mu=0$ and different σ



Normal log-returns



Assume that R is a return on an asset and $X = \ln(1 + R)$ its log-return.

Assume that $X \sim N(\mu, \sigma)$. Then

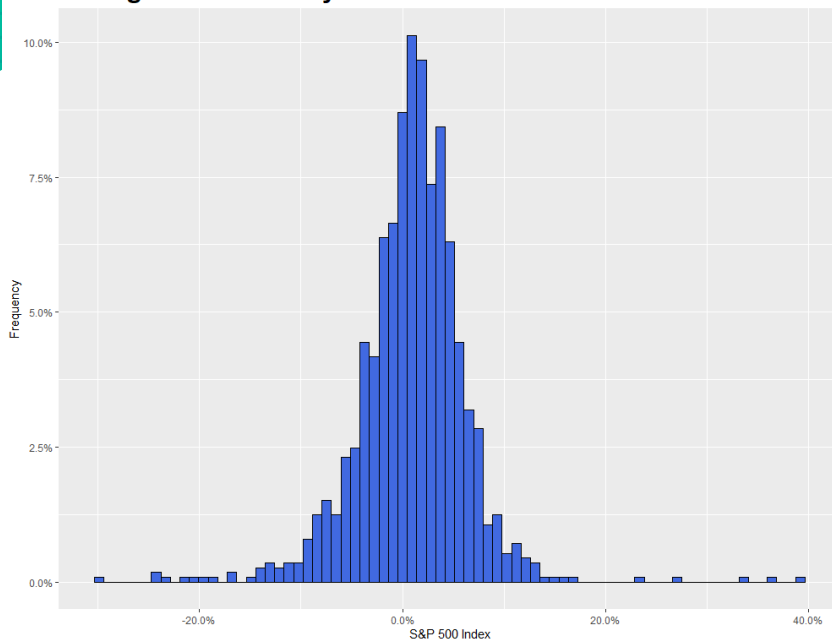
$$R = \exp(X) - 1$$

is distributed according to (shifted) log-normal distribution with parameters μ and σ .

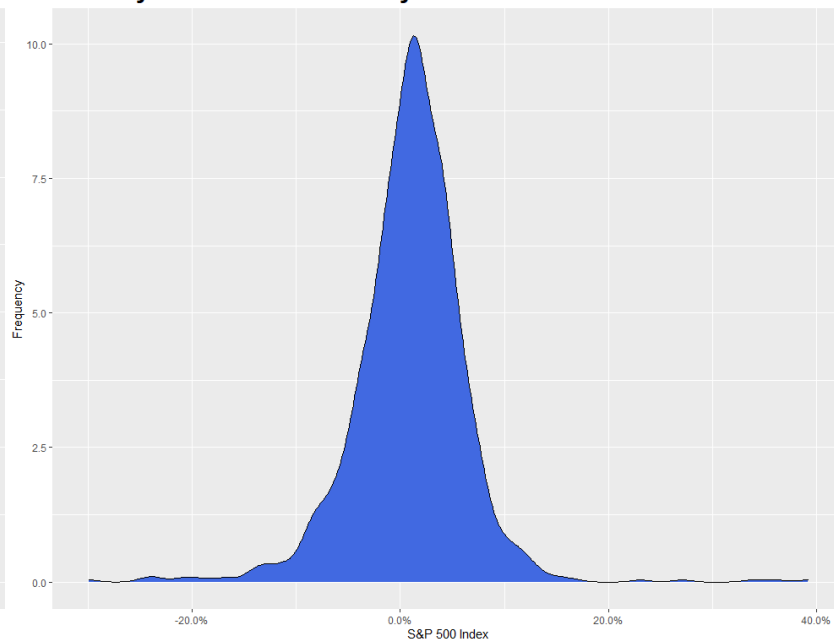
Basic asset models (like Black-Scholes) assume log-normal distribution of asset price

Monthly logreturns on S&P500 Index

Histogram of monthly returns on S&P500

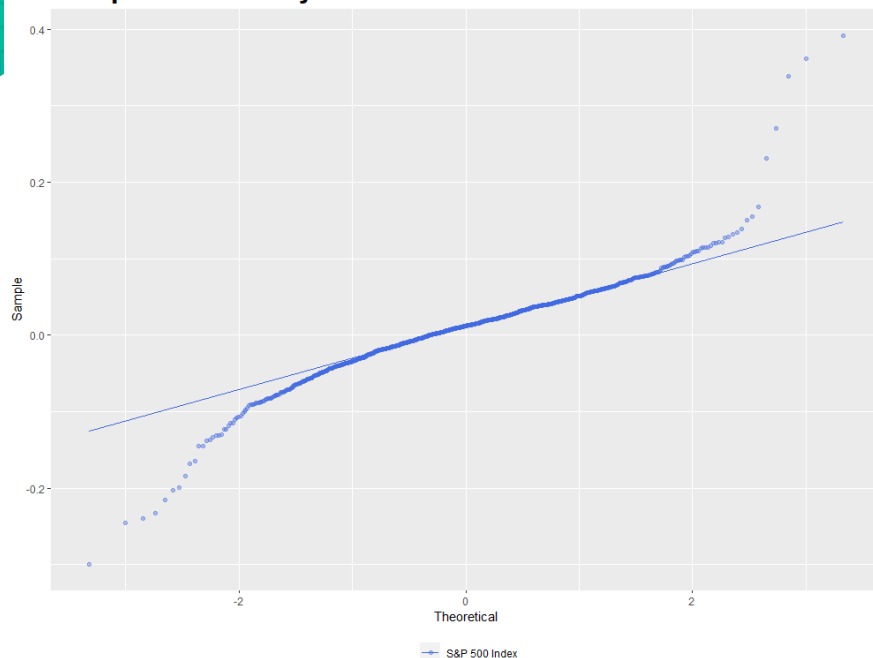


Density estimate of monthly returns on S&P500



Monthly logreturns on S&P500 Index

Q-Q plot of monthly returns on S&P500



0.05-quantile	0.25-quantile	median	mean	0.75-quantile	0.95-quantile	St.dev
-7.6%	-1,6%	1,2%	0,9%	3,8%	8.0%	5,4%

- Clear sign of non-normality
- Negative skew (i.e. distribution is asymmetric and left tail is thicker and median is higher than mean)
- Fat tails

t-distribution



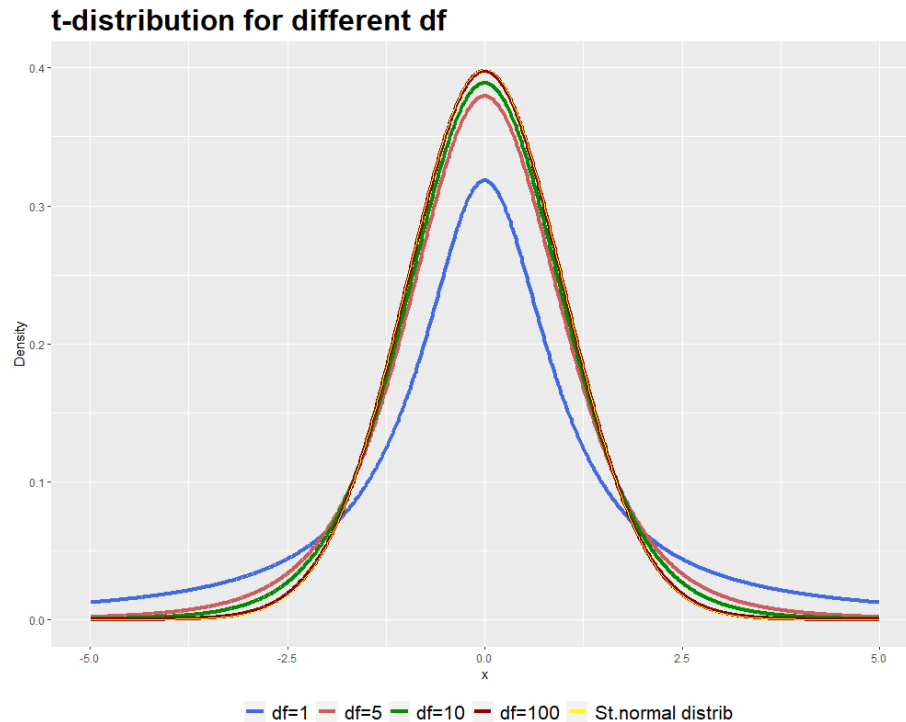
- Normality of log-returns problematic, due to tails (and asymmetry)
- Partial solution: a distribution with heavier tails like Student's t-distribution,
- In the book: t-distribution defined via Gamma-distribution.
- Let $G \sim \text{Gamma}(\alpha, \alpha)$, i.e. shape α and rate α such that mean = 1
- Then if $Z \sim N(0,1)$ and G are two independent random variables, then $X = \frac{Z}{\sqrt{G}}$ is distributed according to t-distribution with α -degrees of freedom
- $E(X) = 0, sd(X) = \sqrt{\frac{\alpha}{\alpha-2}}$. $\alpha > 2$ (error in the book!!!!), excess kurtosis = $\frac{6}{\alpha-4}, \alpha > 4$
- If $\alpha \rightarrow \infty$ then t-distribution converges to normal distribution
- In R: pt, dt, qt, rt from library mvtnorm

T-distribution



- More common, via χ^2 -distribution.
- If $X_1, X_2, \dots, X_K \sim N(0,1)$ i. i. d. Then $\sum_{i=1}^K X_i^2$ has χ^2 -distribution with K-degrees of freedom (denoted $\chi^2(K)$).
- If $Z \sim N(0,1)$ and $S \sim \chi^2(K)$ are two independent random variables then $\frac{Z}{\sqrt{S/K}}$ has t-distribution with K-degrees of freedom
- Easy connection with the test statistics for normal distribution for mu if sigma is unknown

Effect of the parameter change



Returns based on t-distribution

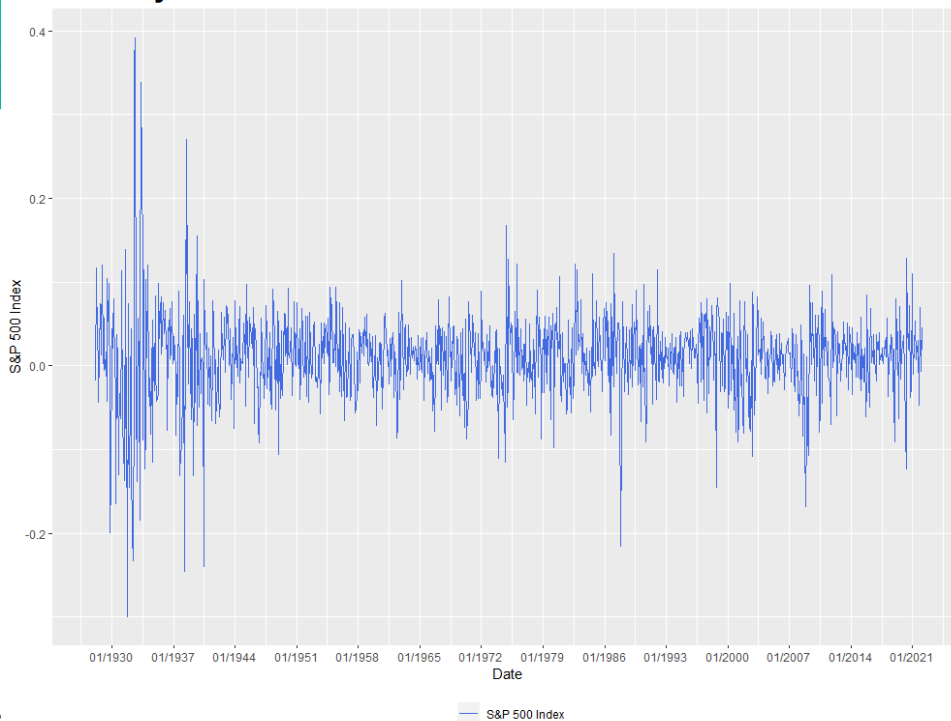
- Let $\mu \in \mathbb{R}, \xi_\sigma > 0, Z \sim N(0,1), G \sim \text{Gamma}(\alpha, \alpha)$ – independent
- Define return R as shifted and scaled t-distribution, i.e.:

$$R = \mu + \xi_\sigma \frac{Z}{\sqrt{G}} = \mu + \sigma Z$$

- $\sigma := \frac{\xi_\sigma}{\sqrt{G}}$ can be seen as **stochastic volatility** (volatility=variability/risk)

Monthly logreturns on S&P500 Index

Monthly returns on S&P500



Characteristics:

- Non-constant and stochastic volatility
- Volatility clustering, i.e. one observes periods of both low and high volatility
- Leverage effect, i.e. high volatility coincides with negative returns
- If we assume that log-returns come from t-distribution, we will miss the two last characteristics
- Solution (way outside this course): model volatility over time as a mean-reverting model

Distributions with non-negative values



Book, chapter 2, section 2.5

Gamma distribution with parameters α and β $\Gamma(\alpha, \beta)$



- Very important and showing up in many places
- Density function: $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$
- $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) dx$ – generalisation of the factorial
- Expected value: $\frac{\alpha}{\beta}$
- Variance: $\frac{\alpha}{\beta^2}$
- Excess kurtosis $\frac{6}{\alpha}$
- In R: `pgamma`, `rgamma`, `dgamma`, `qgamma`

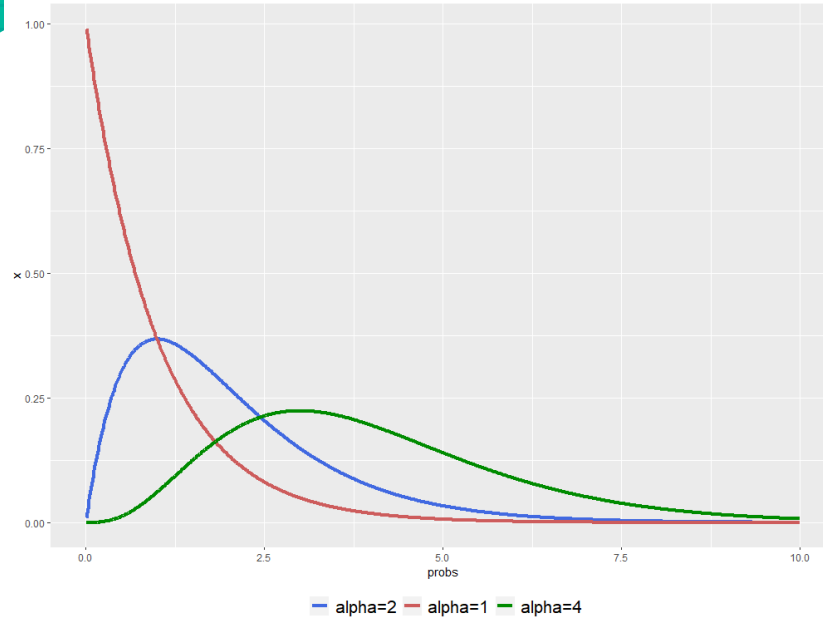
Properties



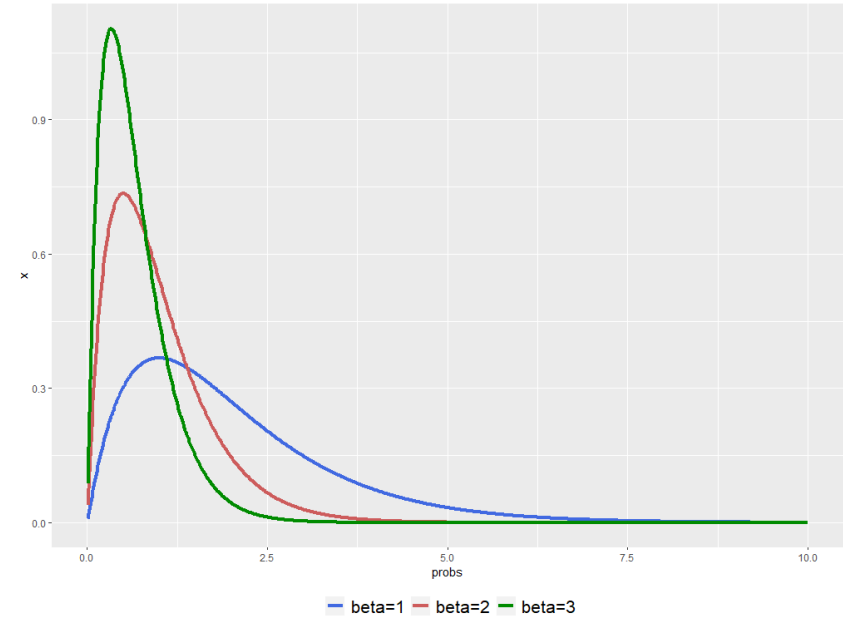
- Let $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$ are two independent random variables then
$$X + Y \sim \Gamma(\alpha_1 + \alpha_2, \beta)$$
- Let $X \sim \Gamma(\alpha, \beta)$ and $c > 0$, then $cX \sim \Gamma(\alpha, \beta/c)$
- If $\alpha = 1$ we get an exponential distribution, which is very important.
- CDF can be nasty in general through so-called incomplete gamma functions.
- Recommendation to use the ready-programmed samplers.

Effect of the parameter change

Gamma density for different alpha, beta=2



Gamma density for different beta, alpha=2

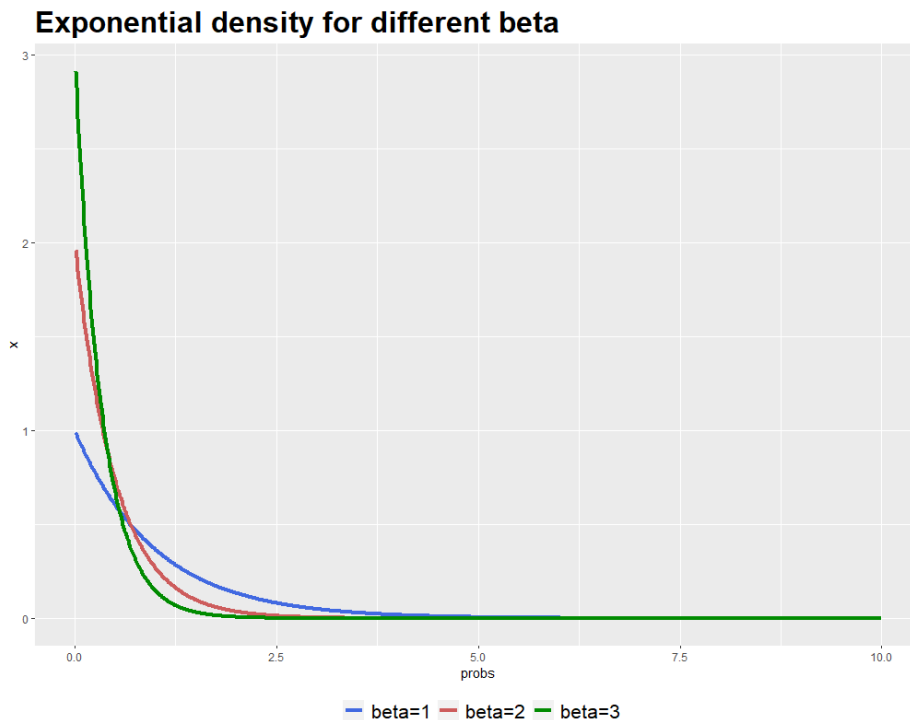


Special case of Gamma: exponential



- If $\alpha = 1$ then Gamma distribution turns into exponential distribution
- Density function: $f(x) = \beta \exp(-\beta x)$
- CDF $F(x) = 1 - \exp(-\beta x)$
- Expected value: $\frac{1}{\beta}$
- Variance: $\frac{1}{\beta^2}$
- Excess kurtosis: 6
- In R: `pexp, dexp, qexp, rexp`
- Important property: **memoryless**: $P(X > a + b | X > b) = P(X > a)$

Effect of the parameter change



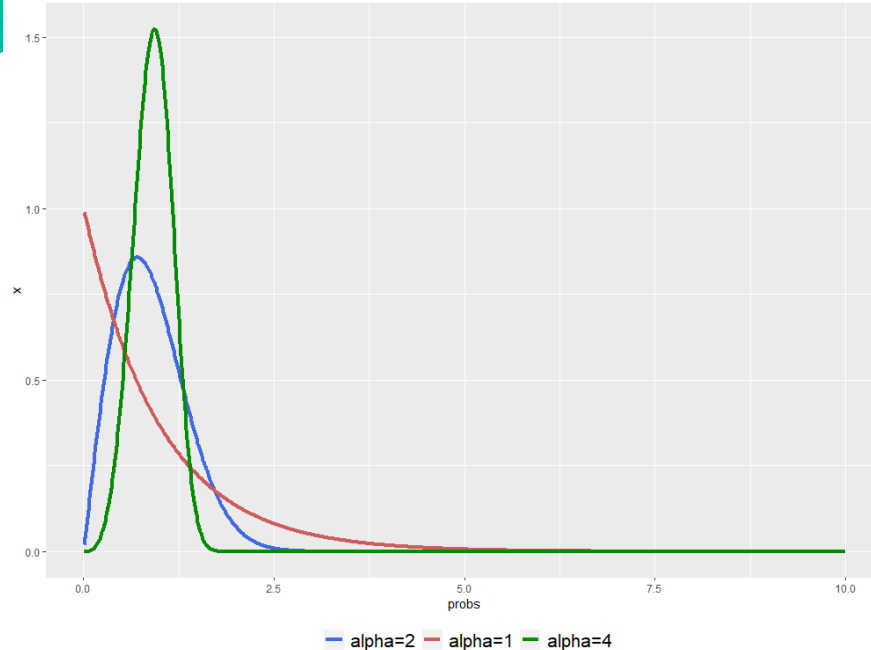
Weibull distribution



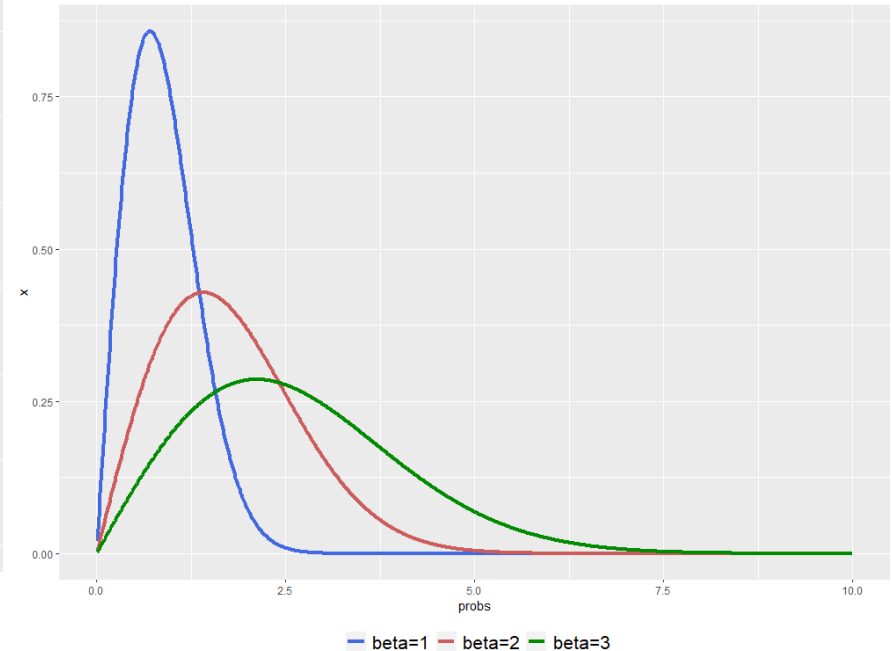
- If $Y \sim \text{Exp}(1)$ then $X = \beta Y^{\frac{1}{\alpha}}$ is Weibull distributed
- Density function $f(x) = \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$
- CDF $F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$
- $E(X) = \beta \Gamma\left(1 + \frac{1}{\alpha}\right)$
- $\text{var}(X) = \beta^2 \left(\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}\right)^2\right)$
- In R: dweibull, rweibull, pweibull, qweibull

Effect of the parameter change

Weibull density for different alpha,beta=2



Weibull density for different beta,alpha=2



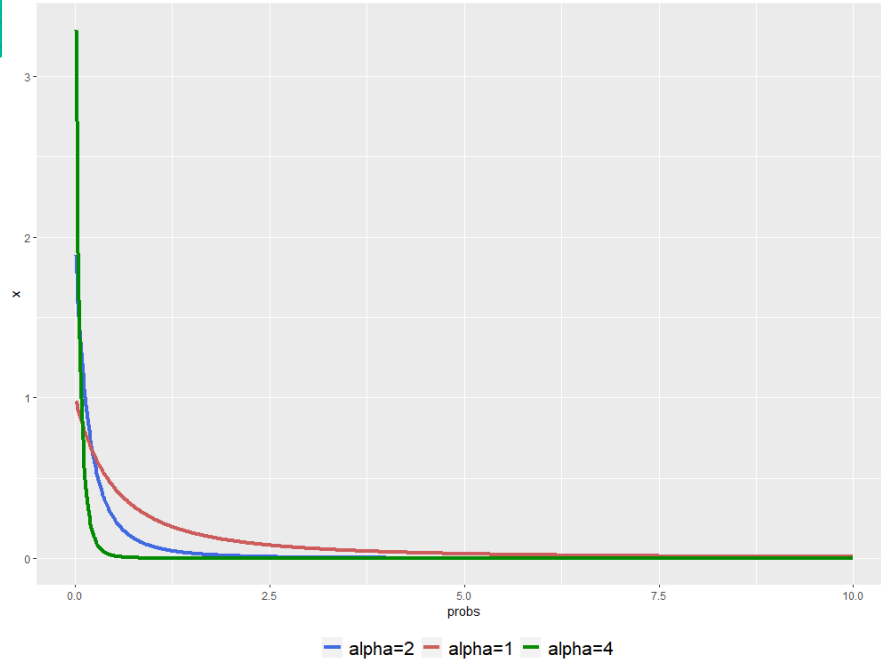
HEAVY tails: Pareto distribution



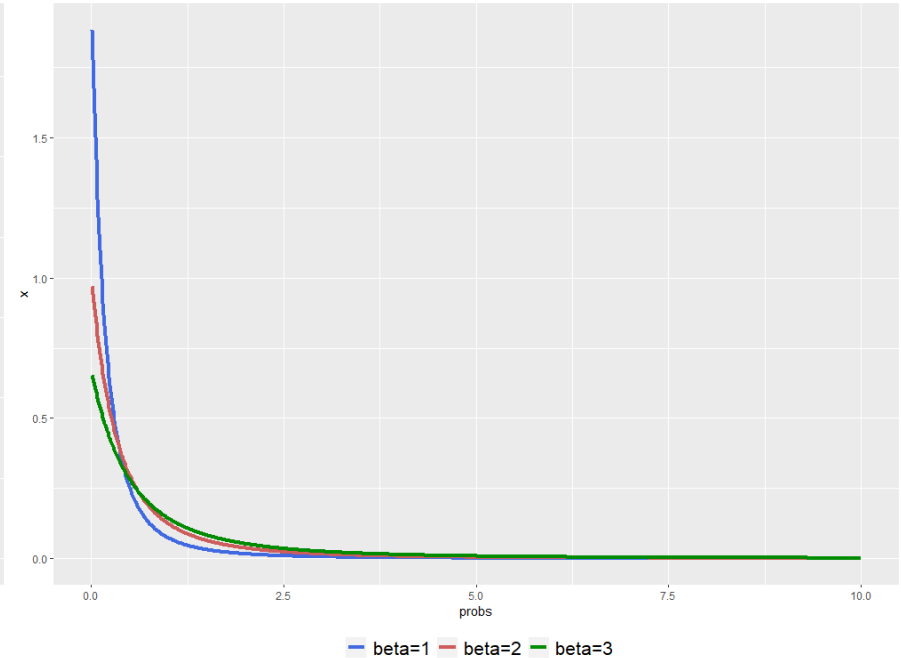
- Density function $f(x) = \frac{\alpha/\beta}{\left(1+\frac{x}{\beta}\right)^{1+\alpha}}$
- CDF $F(x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}$
- $E(X) = \frac{\beta}{\alpha-1}$
- $\text{var}(X) = \frac{E(X)\alpha}{\alpha-2}$
- Sampling through inverse method, or `dpareto`, `ppareto`, `qpareto`, `rpareto` from `EnvStats` – note different parametrization

Effect of the parameter change

Pareto density for different α , $\beta=2$



Pareto density for different β , $\alpha=2$





Discreet distribution: Poisson

- ① $X \sim Poiss(\lambda)$ then
- ① $P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, k=0,1,2,\dots$
- ① $E(X) = var(X) = \lambda$
- ① Properties: a limit distribution of Bernoulli distribution with n trials and $p_n = \frac{\lambda}{n}$.
- ① Therefore law of rare events: Bernoulli(n, p) is close to Poiss(np) for very large n and very small p
- ① Relation with exponential: If $Y_1, Y_2, \dots, Y_n, \dots \sim Exp(\lambda)$ i. i. d., $X \sim Poiss(\lambda t)$ then
- ① $P(X = k) = P(\sum_{i=1}^k Y_i \leq t < \sum_{i=1}^{k+1} Y_i)$
- ① Used to model number of claims in the non-life insurance

Discreet distribution: Negative Binomial

- Derived through Bernoulli trials
- Models the number of failures in a sequence of i.i.d. Bernoulli trials before a specified (non-random) number of successes (denoted r) occurs
- $X \sim NB(r, p)$ then $P(X = k) = \binom{k+r-1}{k} (1-p)^k p^r$
- $E(X) = \frac{pr}{1-p}$
- $\text{var}(X) = \frac{pr}{(1-p)^2}$
- Used to model number of claims in the non-life insurance