

(2.28)

Weibull $X = \beta Y^{1/\alpha}$

$Y \sim \text{Exp}(1)$

where $\alpha > 0, \beta > 0$.

a) F distribution of X is

$$F(x) = P(Y \leq (x/\beta)^\alpha) = 1 - e^{-(x/\beta)^\alpha}$$

b) Show Weibull sampler in Algorithm 2.12 is the inverse sampler.

a) $Y \sim \text{Exp}(1), G(y) = P(Y \leq y) = \int_0^y e^{-u} du$
 $= -[e^{-u}]_0^y = 1 - e^{-y}.$

b) $p = F(x)$

$$\Leftrightarrow p = 1 - e^{-(x/\beta)^\alpha}$$

$$\Leftrightarrow e^{-(x/\beta)^\alpha} = 1 - p$$

$$\Leftrightarrow \left(\frac{x}{\beta}\right)^\alpha = -\log(1-p)$$

$$\Leftrightarrow x = \beta (-\log(1-p))^{1/\alpha}$$

$$\Leftrightarrow F^{-1}(p) = \beta (-\log(1-p))^{1/\alpha}$$

Algorithm 2.12

to pick x, β

sample $U^* \sim \text{Unif}[0,1]$

Return $x^* \leftarrow \beta (-\log U^*)^{1/\alpha}$

Proof follows
from noting
that

$U \sim \text{Unif}[0,1]$

$\Leftrightarrow 1-U \sim \text{Unif}[0,1]$