

3.25

Buyer put option expiring at time T is guaranteed $V_0(1+r_g)$ for equity worth $S_0 = V_0$ at time 0.

Suppose the underlying stock is valued as S_1 at time $T_1 < T$.

a) Argue the put option at T_1 corresponds to a guarantee r_{g1} determined by the equation $(1+r_{g1})S_1 = (1+r_g)V_0$

b) Use Black-Scholes formula

$$\pi(V_0) = \{(1+r_g)e^{-rT} \Phi(a) - \Phi(a - \sigma\sqrt{T})\} V_0$$

$$a = \frac{\ln(1+r_g) - rT + \sigma^2 T/2}{\sigma\sqrt{T}}$$

a) The condition $(1+r_{g1})S_1 = (1+r_g)V_0$ ensures that the original option has the same payoff as a put option initiated at time T_1 with guarantee r_{g1} .

If the price of these two options at time T_1 is different, I can sell the expensive one to buy the cheap one (perfect hedge at time 0) and make a riskless profit with the remainder.

b) Using B-S formula one gets that the price of the old option at time T_1 coincides with the price of a put option with time to expiry $T_e = T - T_1$, guarantee r_2 , and initial price S_1 .

$$\pi_1 = \left\{ (1 + r_2) e^{-r_2 T_e} \Phi(a_1) - \Phi(a_1 - \sigma \sqrt{T_e}) \right\} S_1$$

$$a_1 = \frac{\log(1 + r_2) - r_2 T_e + \sigma^2 T_e / 2}{\sigma \sqrt{T_e}}$$

(3.28)

$$X_p = \max(r_f - R, 0) v_0$$

$$X_c = \max(R - r_f, 0) v_0$$

Payoff Call
and put options.

a) Argue that $X_c - X_p = (R - r_f) v_0$ and, when
viewed as a derivative, priced as
$$e^{-rt} E_Q[X_c - X_p]$$

b) Deduce option premia $\pi_c(v_0)$ and $\pi_p(v_0)$
satisfies

$$\pi_c(v_0) = \pi_p(v_0) + \{L - e^{-rt}(L + r_f)\} v_0$$

c) Using Black-Scholes formula deduce.

$$\pi_c(v_0) = \{ \Phi(-a + \sigma\sqrt{T}) - (L + r_f)e^{-rt}\Phi(-a) \} v_0$$

$$\begin{aligned} \text{a)} \quad X_c - X_p &= \max(R - r_f, 0) v_0 - \max(r_f - R, 0) v_0 \\ &= \begin{cases} (R - r_f) v_0 & \text{if } R \geq r_f \\ -(r_f - R) v_0 = (R - r_f) v_0 & \text{if } R < r_f. \end{cases} \end{aligned}$$

$$\begin{aligned} \text{b)} \quad \pi_c(v_0) - \pi_p(v_0) &= E_Q[e^{-rt} X_c] - E_Q[e^{-rt} X_p] \\ &= e^{-rt} E_Q[X_c - X_p] = e^{-rt} E_Q[(R - r_f) v_0] \\ &= e^{-rt} (E_Q[R] - r_f) v_0 = (*) \end{aligned}$$

But under Q , R grows in average like

The number asset, i.e. $E_Q[R] = e^{rT} - 1$

$$V = \{1 - e^{-rT}(1+r_f)\} v_0 \quad \checkmark$$

c) We have that

$$T_c(v_0) = T_p(v_0) + \{1 - e^{-rT}(1+r_f)\} v_0$$

$$= \{1+r_f\} e^{-rT} \Phi(a) - \Phi(a - \sigma\sqrt{T}) \} v_0 + \{1 - e^{-rT}(1+r_f)\} v_0$$

$$= v_0(1+r_f) e^{-rT} (\Phi(a) - 1) + v_0(1 - \Phi(a - \sigma\sqrt{T}))$$

$$= -v_0(1+r_f) e^{-rT} (1 - \Phi(a)) + v_0 \Phi(\sigma\sqrt{T} - a)$$

$$= v_0 \{ \Phi(\sigma\sqrt{T} - a) - v_0(1+r_f) e^{-rT} \Phi(-a) \} \quad \checkmark$$

3.29 Let $X_p(r)$ and $X_c(r)$ be the payoffs for put and call options with the parameters made explicit. and let $\pi_p(r)$ and $\pi_c(r)$ be its premia.

a) Argue that the payoff

$$H(R) = \begin{cases} r_g - R & R \leq r_g \\ 0 & r_g < R \leq r_c \\ -(R - r_c) & R > r_c \end{cases} \quad \left(\text{Cliquet} \right)$$

$r_g < r_c$

can be written $X = X_p(r_g) - X_c(r_c)$

b) Deduce the premium $\pi = \pi_p(r_g) - \pi_c(r_c)$

or

$$\pi = \pi_p(r_g) - \pi_p(r_c) - \{1 - e^{-rT}(1 + r_c)\}V_0$$

a) If $R \leq r_g$

$$X = \max(r_g - R, 0) - \max(R - r_c, 0)$$

$$= r_g - R - 0 = r_g - R$$

If $r_g < R \leq r_c$

$$X = \max(r_g - R, 0) - \max(R - r_c, 0) = 0 - 0 = 0$$

If $R > r_c$

$$X = \max(r_g - R, 0) - \max(R - r_c, 0) = 0 - (R - r_c)$$

b) Trivial applying linearity of expectation and the put-call parity 3.28 b).