

7.8

Let $z_1, \dots, z_n$ data from

$$f(z) = \frac{1}{\gamma} e^{-z/\gamma} \quad z > 0 \quad \text{Exponential}$$

a) $\hat{\gamma} = \frac{\sum_{i=1}^n z_i}{n}$ moment estimate for γ

b) Show that the log-likelihood function is

$$\ell(\gamma) = -n \log(\gamma) - \gamma^{-1} (z_1 + \dots + z_n)$$

and that is maximized by $\hat{\gamma}$.

a) Let $z \sim f(z)$. Then

$$\begin{aligned} E[z] &= \int_0^{\infty} \frac{z}{\gamma} e^{-z/\gamma} dz = \left[-\cancel{e^{-z/\gamma}} \right]_0^{\infty} + \int_0^{\infty} e^{-z/\gamma} dz \\ &= \int_0^{\infty} \frac{1}{\gamma} e^{-z/\gamma} dz = 1. \end{aligned}$$

Moment of moment, $E[z] = \frac{1}{n} \sum_{i=1}^n z_i$

$$\Rightarrow \hat{\gamma} = \frac{1}{n} \sum_{i=1}^n z_i$$

b) $\ell(\gamma) = \log\left(\prod_{i=1}^n f(z_i)\right) = \sum_{i=1}^n \log f(z_i)$

$$= \sum_{i=1}^n \log \left(\frac{1}{\eta} e^{-t_i/\eta} \right)$$

(2)

$$= \sum_{i=1}^n \log(1/\eta) + \sum_{i=1}^n \log(e^{-t_i/\eta})$$

$$= -n \log(\eta) - \frac{1}{\eta} \sum_{i=1}^n t_i$$

$$\frac{\partial l(\eta)}{\partial \eta} = 0 \Rightarrow -\frac{n}{\eta} + \frac{1}{\eta^2} \sum_{i=1}^n t_i = 0$$

$$\Leftrightarrow \frac{1}{\eta} \sum t_i = n \Leftrightarrow \eta = \frac{\sum t_i}{n}$$

$$\frac{\partial^2}{\partial \eta^2} l(\eta) = \frac{n}{\eta^2} - \frac{2}{\eta^3} \sum t_i =$$

$$\frac{\partial^2}{\partial \eta^2} l(\eta) \Big|_{\eta = \hat{\eta}} = \frac{n}{\hat{\eta}^2} - \frac{2n}{\hat{\eta}^3} \frac{\sum t_i}{n} = -\frac{n}{\hat{\eta}^2} < 0$$

$\Rightarrow \hat{\eta}$ maximum

7.1c

$$Z \sim f(z) = \frac{\alpha}{\beta} \left(\frac{z}{\beta}\right)^{\alpha-1} e^{-\left(\frac{z}{\beta}\right)^\alpha} \quad z > 0$$

a) $z_1, \dots, z_n$ obs.

$$l(\alpha, \beta) = n \log(\alpha) + (\alpha-1) \sum_{i=1}^n \log(z_i) - n\alpha \log(\beta) - \frac{1}{\beta^\alpha} \sum_{i=1}^n z_i^\alpha$$

b) Verify that

$$\frac{\partial l(\alpha, \beta)}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{\alpha}{\beta^{\alpha+1}} \sum_{i=1}^n z_i^\alpha$$

which is zero at $\hat{\beta}_\alpha = \left(\frac{1}{n} \sum_{i=1}^n z_i^\alpha \right)^{1/\alpha}$

c) Argue that the likelihood estimate $\hat{\alpha}$ maximizes $l(\alpha, \hat{\beta}_\alpha)$ with respect to α .

a) clearly.

$$L(\alpha, \beta) = \left(\frac{\alpha}{\beta}\right)^n \prod_{i=1}^n \left(\frac{z_i}{\beta}\right)^{\alpha-1} e^{-\left(\frac{z_i}{\beta}\right)^\alpha}$$

or

$$l(\alpha, \beta) = \log(L(\alpha, \beta)) = \log\left(\left(\frac{\alpha}{\beta}\right)^n\right) + \log\left(\prod_{i=1}^n \left(\frac{z_i}{\beta}\right)^{\alpha-1} e^{-\left(\frac{z_i}{\beta}\right)^\alpha}\right)$$

$$= n \log \alpha - n \log \beta + \log\left(\prod_{i=1}^n \left(\frac{z_i}{\beta}\right)^{\alpha-1}\right) + \log\left(\prod_{i=1}^n e^{-\left(\frac{z_i}{\beta}\right)^\alpha}\right)$$

$$\begin{aligned}
&= n \log \alpha - n \log \beta + (\alpha-1) \sum_{i=1}^n \log(z_i) - (\alpha-1) \sum_{i=1}^n \log(\beta) - \sum_{i=1}^n \left(\frac{z_i}{\beta}\right)^\alpha \\
&= n \log(\alpha) + (\alpha-1) \sum_{i=1}^n \log(z_i) - n \alpha \log(\beta) - \frac{1}{\beta^\alpha} \sum_{i=1}^n (z_i)^\alpha
\end{aligned}$$

$$\begin{aligned}
b) \quad \frac{\partial h(\alpha, \beta)}{\partial \beta} &= 0 + 0 - \frac{n \alpha}{\beta} - \frac{(-\alpha)}{\beta^{\alpha+1}} \sum_{i=1}^n z_i^\alpha \\
&= -\frac{n \alpha}{\beta} + \frac{\alpha}{\beta^{\alpha+1}} \sum_{i=1}^n z_i^\alpha
\end{aligned}$$

$$\frac{\partial h(\alpha, \beta)}{\partial \beta} = 0 \quad (\Leftrightarrow) \quad -\frac{n \alpha}{\beta} + \frac{\alpha}{\beta^{\alpha+1}} \sum_{i=1}^n z_i^\alpha = 0$$

$$(\Leftrightarrow) \quad -1 + \frac{1}{n \beta^\alpha} \sum_{i=1}^n z_i^\alpha = 0$$

$$(\Leftrightarrow) \quad \beta^\alpha = \frac{1}{n} \sum_{i=1}^n z_i^\alpha \quad \Leftrightarrow \quad \beta = \left(\frac{1}{n} \sum_{i=1}^n z_i^\alpha \right)^{1/\alpha}$$

c) $\hat{\alpha}$ i.s. Wert

$$L(\hat{\alpha}, \hat{\beta}) \geq L(\alpha, \hat{\beta}) \quad \forall \alpha, \beta$$

In particular

$$L(\hat{\alpha}, \hat{\beta}) \geq L(\alpha, \hat{\beta}) \quad \forall \alpha$$