

Differences between damage and paid out claims

Insurance companies may cover only a part of the damage that is insured.

The specifics of what is covered and to what degree is stated in the terms and conditions of the contract. In the book, it is defined via a function H such that $\text{Claim} = H(\text{Damage})$

Some important related concepts

- Deductible: The part of the damage that is covered by the insured. Why deductibles?

- * Limits moral hazard, i.e., that the behaviour of the insured changes with insurance.

- * With deductibles, an insured still has a financial stake in preventing the damage (skin in the game)

- * Claim handling costs money: Deductibles make reporting very small claims pointless.

Sum insured: maximum loss that is covered by an insurance company. Why?

- * Sets a clear limit for the liability of an insurance company.

- * Example: marine insurance and oil spill can be very expensive.

Exclusions: Risks which are not covered at all.
Why?

* Risks which are very difficult to insure or not insurable at all

• Example: War

- Can cause a lot of large claims for property or life with high probability.
- Impossible to send a claim adjuster to assess the damage and construction team to repair it.

The deductible, sums insured and exclusions must be taken into account when modelling claim frequency and severity. However, modelling in practice usually occurs on the historical claim data, not the damage data.

This can be a problem in real life (appropriateness of data) but for now on we assume we work on claim data and we ignore such caveat.

Examples:

- Property insurance:

Deductible 4000 kr

Sums insured 10 000 000 kr

Exclusions: Gross negligence (G.N.), War

$$\text{Claim} = \min \left(\max \left(\text{Damage} - 4000, 0 \right), 10\,000\,000 \right) \mathbb{1}_{\{\text{Not War, Not G.N.}\}}$$

- Pet insurance (medical expenses)

Deductible: 20% expenses, but at least 1500 kr.

Sums insured: 30 000 kr

Exclusions: Medical expenses not caused by illness or accident

$$\text{Claim} = \min \left(\max \left(\text{Damage} - \max(0.2 \times \text{Damage}, 1500), 0 \right), 30\,000 \right) \mathbb{1}_{\{\text{Illness or accident}\}}$$

Reinsurance

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Insurance companies might want to insure same risks, but then move some part of the risk (cedes the risk) to another company - reinsurance company.

There are big global companies that use the same type of argument (LLD) to reinsure and diversify risks that regular or small insurance companies do not want to keep.

In practice, insurance companies cedes a part of premiums by paying a reinsurance premium to a reinsurance company, while reinsurance companies compensate the insurance companies for all the claims that qualify.

Reinsurance can occur either per event or per portfolio. That is, the reinsurance can take over a part of risk for single events (e.g. large claims) in a portfolio or a part of general risk in a portfolio (e.g. the portfolio has generated too much losses in total).

Some examples of risk that are usually reinsured:

- Catastrophe risks - Both NatCat (Natural catastrophes) and Man-Made.
- Non-Cat large claims.

It is also common to find reinsurance between companies within a group (internal reinsurance)

This can be explained by management / tax optimisation perspective.

Some important definitions and notation:

Risk / premium ceded: risk or premiums which are moved to a reinsurance company.

Reinsurance premium: what the insurance company pays to the reinsurance company, which usually coincides with the premiums ceded.

Pure reinsurance premium: part of the reinsurance premium due to the amount needed to cover expected losses and loss adjustment expenses, but excluding overhead costs, administrative fees or profit margins for the reinsurer.

Risk retained: the risk left to the insurance company after the reinsurance contract.

Reinsurance recoverables: part of claims which the reinsurance company is responsible for.

Gross / Net claims: claim amounts before / after reinsurance recoverables.

Gross / Net premiums: premiums before / after deducting reinsurance premiums.

Notation

Let $X = \sum_{i=1}^N \varepsilon_i$ be a sum of claim amounts before reinsurance.

Let H^{event} be a function which calculates reinsurance recoverable for a contract per event. Then,

$X^{\text{re}} = \sum_{i=1}^N H^{\text{event}}(\varepsilon_i)$ is the sum of reinsurance

recoverables from the contract.

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Let $H^{\text{portfolio}}$ be a function which calculates reinsurance recoverables for a portfolio

then $X^{\text{re}} = H^{\text{portfolio}}\left(\sum_{i=1}^N Z_i\right)$ is a sum of reinsurance recoverables from the portfolio.

Net claims are then denoted as $X^{\text{ce}} = X - X^{\text{re}}$.

Types of reinsurance

Proportional

An insurance company cedes a predefined percentage of each premium from a specific line of business to the reinsurance company, and the reinsurance company pays the same percentage of each claim.

$$H^{\text{net}}(x) = H^{\text{portfolio}}(x) = \gamma x$$

This method is useful when a company does not have enough capital to write new business. It is also useful for capital optimisation within a group: smaller daughter companies are less diversified than the parent company and therefore require more capital. Reinsurance can move the risk to the parent.

Non-proportional

There are many variations of this method but in principle is used to transfer **tail risk**

to the reinsurance company.

This is done to protect capital from the impact of large rare events or cumulation of rare events. You get to optimize the capital reserved by the insurance company and reduce the volatility.

Examples:

Excess-of-loss per event (layer $A \times B$).

Insurance company pays a fixed premium, and reinsurance company pays the amount exceeding A , but up to B for each claim.

Stop loss

This contract works on the portfolio level. Reinsurer covers all the claims (regardless size) when the total amount of claims so far has exceeded a predefined threshold. (usually % of premiums)

Example: We have written 100 million K€ in premiums and we have 110% stop loss contract. Then the reinsurance company will cover all new claims from the moment our cumulative claims reach the level 110 million K€.

Pricing of an insurance contract

So far we have assumed a compound Poisson model, either on portfolio basis or on a policy basis, and we calculate the pure premium based on it.

To model at policy level, we assume that the intensity of the Poisson process and the parameters of the claim severity distribution may depend on the policy.

How is this dependency?

Assume we suspect that some variables can determine the frequency of claims, e.g., a young driver tends to drive faster than an older one and causes more accidents. We call such a variable, "age" in this example, a **risk factor**.

Risk differentiation is a process to determine such risk factors and divide the portfolio into small groups with similar risks.

These groups will get the same pure premium.

Risk differentiation can be done both for the claim frequency and claim size.

We will focus on claim frequency.

Motivation for the model

Recall the exponential family of distributions, that is, the distributions (discrete and continuous).

for which the probability function can be written as 8

$$f_{\theta}(y) = h(y) \exp(\eta(\theta) \cdot T(y) - A(\theta))$$

where $\theta \in \mathbb{R}$ is the parameter of the family, h , η , T and A are known functions.

Many known distributions can be written in this form.

If η and T are the identity functions, then the family is called a natural exponential family. We assume from now on that $\eta(\theta) = \theta$ and $T(y) = y$.

Moreover, if $X \sim f_{\theta}$ then the moment generating function is given by

$$M_X(t) = \exp(A(\theta + t) - A(\theta))$$

$$\Rightarrow E[X] = \left. \frac{dM_X(t)}{dt} \right|_{t=0} = M_X(t) \cdot A'(\theta + t) \Big|_{t=0}$$

$$= \underbrace{M_X(0)}_1 A'(\theta)$$

$$E[X^2] = \left. \frac{d^2 M_X(t)}{dt^2} \right|_{t=0} = M_X(t) (A'(\theta + t))^2$$

$$+ M_X(t) A''(\theta + t) \Big|_{t=0} = (A'(\theta))^2 + A''(\theta)$$

$$\text{and } \text{Var}[X] = E[X^2] - (E[X])^2 = A''(\theta)$$

For the poisson distribution

$$p_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda} = \frac{1}{k!} \exp(k \log(\lambda) - \lambda) \quad \underline{19}$$

and we get

$$h(y) = \frac{1}{y!}, \quad \theta = \log(\lambda) \text{ and } A(\theta) = e^\theta$$

We assume that our frequency depends on some risk factors $X_1, X_2, \dots, X_n$.

Example: Car insurance $X_1, \dots, X_n$ can be age of the driver, power of the engine, ...

In regular regression we want to model

$$E[Y | X_1, \dots, X_n] = \alpha_0 + \alpha_1 X_1 + \dots + \alpha_n X_n,$$

but, this can yield negative numbers.

However, since for the Poisson we know that

$$E[Y | X_1, \dots, X_n] = \frac{d}{d\theta} A(\theta | X_1, \dots, X_n) = \exp(\theta | X_1, \dots, X_n)$$

and θ can take all real values, we can model θ with linear regression.

Therefore

$$E[Y | X_1, \dots, X_n] = \exp(\theta | X_1, \dots, X_n) = \exp(\alpha_0 + \alpha_1 X_1 + \dots + \alpha_n X_n)$$

One can assume the following model

$$\begin{aligned} E[N | X_1, \dots, X_n] &= \exp(\alpha_0 + \alpha_1 X_1 + \dots + \alpha_n X_n) \\ &= \exp(\alpha_0) \exp(\alpha_1 X_1) \dots \exp(\alpha_n X_n) \end{aligned}$$

where $N \sim \text{Pois}(\lambda)$, $\exp(\alpha_0)$ is a base frequency and $\exp(\alpha_i X_i)$ are loading factors saying how much the actual frequency will fall/increase

compared to the base frequency for a particular¹⁶ value of the risk factor X_i .

Easy to interpret and to check reasonableness of the model.

Typical risk factors:

- Car insurance: age of driver, power of the engine, age of car, experience of the driver, number of claims in the past, value of the car, ...
- Property insurance: age of building, number of storeys, area, construction type, type of roof, number of storeys, underground, number of people in the household, location, number of claims in the past, rented or not, ...

Similar regression can be made for average claim severity. Depending on the claim size distribution we may have different "link" between regressors $X_1, \dots, X_n$ and $E[Z_L | X_1, \dots, X_n]$.

Then, we can provide a premium given the value of the risk factors $X_1, \dots, X_n$

$$\begin{aligned} \pi_{X_1, \dots, X_n}^{P_n} &= E[X | X_1, \dots, X_n] = E[E[X | N, X_1, \dots, X_n] | X_1, \dots, X_n] \\ &= E\left[N \underbrace{E[Z_L | N, X_1, \dots, X_n]}_{E[Z_L | X_1, \dots, X_n]} | X_1, \dots, X_n\right] \end{aligned}$$

$$= E[N | X_1 \dots X_n] E[Z_1 | X_1 \dots X_n]$$

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where we have assumed that, conditionally on $X_1, \dots, X_n$, $N, Z_1, \dots, Z_n$ are ^{still} independent and Z_i are i.i.d.

Another extension of the compound Poisson model

We usually have

Claims = $X = \sum_{i=1}^N Z_i$. N r.v. with values $\mathbb{Z}_+$ and Z_i , i.i.d., independent of N .

Usually, we assume that $N \sim \text{Pois}(\lambda)$ and that we know λ .

What happens if λ is unknown? We get then a model of a C.P. with intensity uncertainty

1. $X = \sum_{i=1}^N Z_i$

2. Z_i , i.i.d.

3. λ is a r.v. independent of Z_i and, given λ the conditional law of $N \sim \text{Pois}(\lambda)$.

4. N is still independent of Z_i

We proved in the previous lecture that if $N, Z_1, Z_2, \dots$ are independent then

$$E[X] = E[N] E[Z_1]$$

and
$$\text{Var}[X] = E[N] \text{Var}[Z_1] + \text{Var}[N] (E[Z_1])^2$$

Since N given λ is $\text{Pois}(\lambda)$

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$$E[N|\lambda] = \lambda, \quad \text{Var}[N|\lambda] = \lambda$$

and

$$E[N] = E[E[N|\lambda]] = E[\lambda]$$

$$\begin{aligned} \text{Var}[N] &= E[\text{Var}[N|\lambda]] + \text{Var}[E[N|\lambda]] \\ &= E[\lambda] + \text{Var}[\lambda] \end{aligned}$$

Hence,

$$E[X] = E[\lambda] E[z_i]$$

$$\begin{aligned} \text{Var}[X] &= E[\lambda] \text{Var}[z_i] + (E[\lambda] + \text{Var}[\lambda]) (E[z_i])^2 \\ &= (\text{Var}[z_i] + (E[z_i])^2) E[\lambda] + (E[z_i])^2 \text{Var}[\lambda] \\ &= E[z_i^2] E[\lambda] + (E[z_i])^2 \text{Var}[\lambda] \end{aligned}$$

Assume now that $\lambda = \gamma \mu^T$, where μ is a r.v. such that λ satisfies the previous model hypothesis. Then

$$\begin{aligned} CV(X) &= \frac{sd(X)}{E[X]} = \frac{\sqrt{\gamma^T E[z_i^2] E[\mu] + \gamma^T +^2 (E[z_i])^2 \text{Var}[\mu]}}{\gamma^T E[z_i] E[\mu]} \\ &= \sqrt{\frac{E[z_i^2] E[\mu]}{\gamma^T (E[z_i])^2 (E[\mu])^2} + \frac{\text{Var}[\mu]}{E[\mu]^2}} \end{aligned}$$

Note that when $\gamma \rightarrow \infty$ $CV(X) \rightarrow CV[\mu]$

In this model the risk per unit exposure can be decomposed into

- a diversifiable risk which can be reduced to zero by increasing the number of policies γ
- an undiversifiable risk which cannot be eliminated and is linked to the uncertainty on the intensity parameter λ .