Introduction to Insurance Mathematics (STK 3505/4505)
Exercises 9

Problem 1 Once more let us cast a glance at the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 09/01/1983 until 09/25/1983:

Date	Loss X_i in DKM	Date	Loss X_i in DKM
09/03/1983	3.876529	09/16/1983	3.061179
09/03/1983	3.857319	09/17/1983	1.436040
09/04/1983	1.437954	09/18/1983	2.279199
09/05/1983	1.175275	09/18/1983	1.491766
09/08/1983	1.501669	09/19/1983	6.916554
09/09/1983	1.506003	09/21/1983	1.604229
09/13/1983	5.925473	09/25/1983	1.668521
09/16/1983	12.631813	09/25/1983	1.157211

The fire losses in the above table are given in millions of Danish Krone. Suppose that the claim size distribution is modelled by an exponential distribution with intensity $\lambda > 0$.

(i) Compute the maximum likelihood estimator $\widehat{\lambda}$ of the intensity of the claim sizes. Sketch the graph of the QQ-plot of the above data against $F \sim Exp(\widehat{\lambda})$.

(ii) Sketch the graph of the ME-plot.

Hint: Apply the following convention: Claims $X_1^{(m)}, ..., X_n^{(m)}$ arriving at day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^n X_i^{(m)}$ arriving at day m. Employ Problem 2 to evaluate $e_{F_n}(X_{(k)})$, $k = 1, ..., n-1$.

Problem 2 Show that the empirical mean excess function $e_{F_n}(u)$ is given by

$$e_{F_n}(u) = \frac{\sum_{i=1}^n \max(X_i - u, 0)}{\#\{i \leq n : X_i > u\}}$$

for all sample sizes $n \geq 1$.# stands for the number of elements in a set.

Problem 3 Let F be the Pareto distribution, that is

$$F(x) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^{\alpha}, \ x > 0$$

for parameters $\alpha, \kappa > 0$.

Verify that the mean excess function of F is given by

$$e_F(u) = \frac{\kappa + u}{\alpha - 1}$$

for $\alpha > 1$, $\kappa > 0$.

Problem 4 Consider two portfolios of fire insurance policies with total claim amounts described by

$$S_i(t) = \sum_{j=1}^{N_i(t)} X_j^{(i)},\ i = 1, 2,$$

where the claim numbers $N_1(t)$ and $N_2(t)$ are homogeneous Poisson processes with intensities $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{2}{3}$, respectively. The claim numbers $N_i(t)$ are assumed to be independent of the $i.i.d$ fire losses $(X_j^{(i)})_{j\geq 1}$ for each fixed $i = 1, 2$. Further suppose that the total claim amounts are independent and that the claim size distributions are specified by

$$\begin{array}{lll} P(X_j^{(1)} = 0) = 0 & P(X_j^{(1)} = 1) = \frac{1}{3} & P(X_j^{(1)} = 3) = \frac{2}{3} \\ P(X_j^{(2)} = 0) = 0 & P(X_j^{(2)} = 2) = \frac{1}{2} & P(X_j^{(2)} = 4) = \frac{1}{2} \end{array}$$

for all $j \geq 1$. The claim size values are expressed in millions of Danish Krone.

Characterize the distribution of the aggregated total claim amounts, that is the distribution of

$$\widetilde{S}(t) = S_1(t) + S_2(t)$$

in terms of mixture distributions for $t = 1$ year.

Problem 5 Let $N(t)$, $t \geq 0$ be claim numbers modelled by a Poisson process with mean value function $\mu(t)$. Further denote by $(X_j^{(i)})$ the claim numbers in year $i = 1, ..., n$. Assume that $(X_j^{(i)})$ is $i.i.d$ for each i and that the claim sizes $X_j^{(i)}$ are mutually independent and independent of N. The total claim amount in year i is given by

$$S_i = \sum_{j=N(i-1)+1}^{N(i)} X_j^{(i)}.$$

Show that

$$\left(\sum_{j=N(i-1)+1}^{N(i)} X_j^{(i)} \right)_{i=1,...,n} \stackrel{d}{=} \left(\sum_{j=1}^{N(i)-N(i-1)} X_j^{(i)} \right)_{i=1,...,n}.$$

Use the latter relation to conclude that S_i, $i = 1, ..., n$ are independent compound sums.