Exercises I

Problem I : perpetuity (i.e. payment stream of infinite duration) with parameters

m # of payments per year

q # of payment increases

$\longrightarrow$ cash flow

time	paym.	
$0 \mid \frac{1}{m} \mid \cdots \mid \frac{1}{q} - \frac{1}{m}$	$\frac{1}{m \cdot q}$	; can be
$\frac{1}{q} \mid \frac{1}{q} + \frac{1}{m} \mid \cdots \mid \frac{2}{q} - \frac{1}{m}$	$\frac{1}{m \cdot q} + \frac{1}{m \cdot q}$	=
$\frac{2}{q} \mid \frac{2}{q} + \frac{1}{m} \mid \cdots \mid \frac{3}{q} - \frac{1}{m}$	$\frac{1}{m \cdot q} + \frac{1}{m \cdot q} + \frac{1}{m \cdot q}$	decomposed into the foll. sum of cashflows
⋮	⋮	

time	paym
0 - -	$\frac{1}{m \cdot q}$
$\frac{1}{q}$ - - -	$\frac{1}{m \cdot q}$
$\frac{2}{q}$ - - -	$\frac{1}{m \cdot q}$
⋮	⋮

perp. starting in 0

+

time	paym
0 - - -	0
$\frac{1}{q}$ - - -	$\frac{1}{m \cdot q}$
$\frac{2}{q}$ - - -	$\frac{1}{m \cdot q}$
⋮	⋮

perp. start. in $\frac{1}{q}$

+

time	paym
0 - - -	0
$\frac{1}{q}$ - - -	0
$\frac{2}{q}$ - - -	$\frac{1}{m \cdot q}$
⋮	⋮

perp. start. in $\frac{2}{q}$

+ ...

$\longrightarrow$ perp. with constant payments $\frac{1}{m \cdot q}$

$$\Rightarrow \left(J^{(q)}\ddot{a}\right)^{(m)}_{\infty|} = \sum_{l \geq 0} \underbrace{\sum_{k \geq 0} \left(\frac{1}{m \cdot q}\right) \left(v^{\frac{1}{q}}\right)^{l} \cdot \left(v^{\frac{1}{m}}\right)^{k}}_{\text{perp. starting in } \frac{l}{q} \text{ with paym. } \frac{1}{m \cdot q}}$$

$$= \left(\sum_{l \geq 0} \frac{1}{q} \left(v^{\frac{1}{q}}\right)^{l}\right) \left(\sum_{k \geq 0} \frac{1}{m} \left(v^{\frac{1}{m}}\right)^{k}\right)$$

$$= \ddot{a}^{(m)}_{\infty|} \cdot \ddot{a}^{(q)}_{\infty|} \overset{\text{Ex. 2.5.3}}{=} \frac{1}{d^{(m)}} \frac{1}{d^{(q)}}$$

$d^{(l)}$ nom. discount rate with conv. period $\frac{1}{l}$

(2)

Problem 2

(i) a) $a_{\overline{n}|} = \sum_{k=1}^{n} v^k = \frac{1 - v^{n+1}}{1-v} - 1 = \frac{1-v}{1-v}$

$= \frac{1 - v^{n+1} - 1 + v}{1-v} = \frac{v - v^{n+1}}{1-v} =$

$\frac{v(1-v^n)}{1-v} = \frac{1}{i}(1-v^n)$, since $v = \frac{1}{1+i}$

b) $\ddot{a}^{(m)}_{\overline{n}|} = \sum_{k=0}^{n \cdot m - 1} \frac{1}{m} \left(v^{\frac{1}{m}}\right)^k =$

$\frac{1}{m} \frac{1 - \left(v^{\frac{1}{m}}\right)^{n \cdot m}}{1 - v^{\frac{1}{m}}} = \left(\frac{1}{m} \cdot \frac{1}{1 - v^{\frac{1}{m}}}\right)(1 - v^n)$

$= \frac{1}{d^{(m)}}$ (Ex. 2.5.3)

c) $a^{(m)}_{\overline{n}|} = \sum_{k=1}^{n \cdot m} \frac{1}{m} \left(v^{\frac{1}{m}}\right)^k = \frac{1}{m} v^{\frac{1}{m}} \left(\sum_{\ell=0}^{n \cdot m - 1} \left(v^{\frac{1}{m}}\right)^\ell \right)$

$= \frac{1}{m} \cdot v^{\frac{1}{m}} \frac{(1-v^n)}{(1-v^{\frac{1}{m}})} = \frac{\frac{1}{m} \cdot v^{\frac{1}{m}}}{(1-v^{\frac{1}{m}})} (1-v^n)$

$= \frac{1}{i^{(m)}}$ $\left(\iff i^{(m)} = m\left(\frac{1}{v^{\frac{1}{m}}} - 1\right)\right)$

(ii) $s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$

$\Rightarrow \frac{1}{s_{\overline{n}|}} + \frac{1}{i} = \frac{i}{(1+i)^n - 1} + i = \frac{i(1+i)^n}{(1+i)^n - 1}$

$= i \cdot \frac{1}{1-v^n} = \frac{1}{a_{\overline{n}|}}$

Problem 3

S_k principal outstanding (i.e. value of remaining debt at $t=k$ without r_k)

→ payments 0 at $t=k$, r_{k+1} at $t=k+1$, ..., r_n at $t=n$

$\Rightarrow$ $S_k = v r_{k+1} + v^2 r_{k+2} + \dots + v^{n-k} r_n$ (prospective formula for S_k) (*)

$S := \sum_{h=1}^{n} v^h r_k$ (PV of the debt)

$\overset{(*)}{\Rightarrow}$, $v = \frac{1}{1+i}$: $S_k = (1+i)^k S - \sum_{h=1}^{k} (1+i)^{k-h} r_h$ (**)

(retrospective formula for S_k, looking at the prev. paym.)

(3.) Use (2.1.2):

$$F_n = (1+i)^n F_0 + \sum_{k=1}^{n} (1+i)^{n-k} r_k$$

Solves the recursion

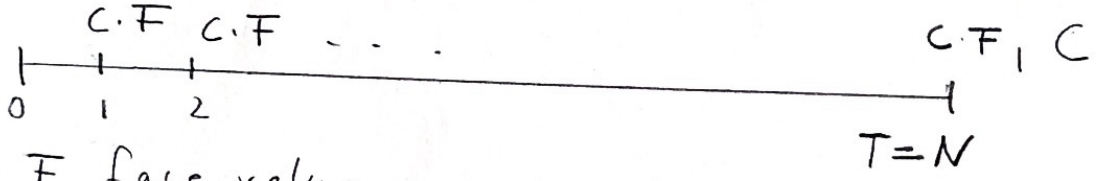
$$r_k = i(-F_{k-1}) + (F_k - F_{k-1}) \quad \text{(see (2.1.1))}$$

Choose $F_k = -S_k \Rightarrow$

$$r_k = \underbrace{i S_{k-1}}_{\text{int. of running debt}} + \underbrace{(S_{k-1} - S_k)}_{\text{red. of principal}}$$

Problem 4:

bondholder (or lender) gets the foll. payment stream from the bond issuer (or borrower)



F face value
c coupon rate
C redemption value

$$\longrightarrow PV = C \cdot F \underbrace{\frac{1-(1+i)^{-N}}{i}}_{= a_{\overline{N}|}} + C(1+i)^{-N}$$

i yield to maturity

Company wants to retire the bond at Jan. 1, 2015
$\Rightarrow$ has to pay the value of the bond at Jan. 1, 2015

$$\rightarrow B = \sum_{k=1}^{15} 15000 \left(\frac{1}{1+0.06}\right)^k = 15000\, a_{\overline{5}|}$$

$i = 0.06,\ C = 0$

For this purpose the company deposits money

$\rightarrow$ value of the deposits (Jan. 1, 2015):

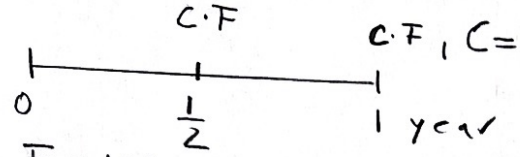
$$D = \sum_{k=0}^{9} X (1+0.06)^k = X \left(\frac{1-(1+0.06)^{10}}{1-(1+0.06)}\right) = X \cdot s_{\overline{10}|}$$

$\rightarrow$ Objective: $B = D$

$$\Rightarrow X = 15000 \frac{a_{\overline{5}|}}{s_{\overline{10}|}} = 4794$$

(4) Problem 5:

bond of value 98.51 has cash flow



$F = 100$, $c = 0.04/2$

$$\Rightarrow 98.51 = PV = \underbrace{\frac{C.F}{=2}}_{=x} \cdot \frac{1}{1+\frac{i^{(2)}}{2}} + \underbrace{\frac{C.F}{=2}}_{=x} \cdot \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^2} + \underbrace{\frac{C}{F}}_{=102} \cdot \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^2}$$

$$= 2\,\frac{1}{1+\frac{i^{(2)}}{2}} + 102\,\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^2}$$

$$\rightarrow x = 0.9729882 \Rightarrow i^{(2)} = 5.55\%$$