Exercises 2

Problem 1 :

Recall that

$$ {}_t p_{x+s} \overset{\text{def.}}{=} \Pr(T > s+t \mid T > s) = \frac{{}_{t+s}p_x}{{}_s p_x} $$

(prob. (x) survives another t years, after having reached $x+s$ years)

$$ {}_u p_x = 1 - G(u) = \Pr(T > u) $$

$$ \Rightarrow \quad p_x \cdot p_{x+1} \cdot p_{x+2} \cdots p_{x+k-1} $$

$$ = {}_1p_x \cdot \frac{{}_2p_x}{{}_1p_x} \cdot \frac{{}_3p_x}{{}_2p_x} \cdots \frac{{}_{k-1}p_x}{{}_{k-2}p_x} \cdot \frac{{}_k p_x}{{}_{k-1}p_x} = {}_k p_x $$

Problem 2 :

(i) a) $ {}_t p_{x+s} \overset{\text{Prob. 1}}{=} \frac{{}_{t+s}p_x}{{}_s p_x} \Rightarrow {}_{t+s}p_x = {}_s p_x \cdot {}_t p_{x+s} $

b) $ {}_{s|t}q_x = G(s+t) - G(s) = \underbrace{[1-G(s)]}_{= {}_s p_x} \underbrace{\frac{G(s+t)-G(s)}{1-G(s)}}_{= \Pr(T \le s+t \mid T > s) \overset{\text{def}}{=} {}_t q_{x+s}} $

(ii) $ {}_s q_{x+t} = \frac{G(s+t)-G(t)}{1-G(t)} $

$ \approx G'(t) = g(t) $ for s small

$$ = \left(\frac{\frac{G(t+s)-G(t)}{s}}{1-G(t)} \right) \cdot s $$

$$ \approx \underbrace{\frac{g(t)}{1-G(t)}}_{= \mu_{x+t}} \cdot s = \mu_{x+t} \cdot s $$

Problem 3 :

$$ {}_t p_x \overset{(3.2.2)}{=} e^{-\int_0^t \mu_{x+s}\, ds} $$

$$ = e^{-\int_0^t \kappa (x+t)^n \, ds} $$

$$ = \exp\left(-\frac{\kappa}{n+1} (x+s)^{n+1} \Big|_{s=0}^{t} \right) $$

$$ = \exp\left(-\frac{\kappa}{n+1} \left[(x+t)^{n+1} - x^{n+1} \right] \right) $$

Problem 6 :

$ K \overset{\text{def.}}{=} [T(x)] $, $ S = T - K \in [0,1) $

(i)

(2)

Let $t \in [0,1]$

$$\Rightarrow \Pr(S \le t) = \Pr\big((T - \underbrace{[T]}_{\text{def. } K}) \le t\big)$$

$$= \Pr\Big(\bigcup_{\ell \ge 0} \underbrace{\{\ell \le T < \ell + t\}}_{=: A_\ell}\Big)$$

$$\overset{A_i \cap A_j = \emptyset,\ i \ne j}{=} \sum_{\ell \ge 0} \Pr(\ell \le T < \ell + t)$$

$$= \sum_{\ell \ge 0} \big(G(\ell + t) - G(\ell)\big)$$

$$= \sum_{\ell \ge 0} \int_{\ell}^{\ell + t} g(s)\, ds = \sum_{\ell \ge 0} \int_{0}^{t} g(\ell + s)\, ds$$

$$\overset{\text{mon. convergence}}{=} \int_0^t \underbrace{\Big(\sum_{\ell \ge 0} g(\ell + s)\Big)}_{=: f \text{ density}} ds$$

(ii) $\mathrm{Var}[T] \overset{\text{def.}}{=} \mathrm{Var}[K + S]$

$$= E[(K + S - E[K + S])^2]$$

$$= E[(\underbrace{K - E[K]}_{=: A} + \underbrace{(S - E[S])}_{=: B})^2]$$

$$= E[A^2 + 2AB + B^2]$$

$$= \underbrace{E[A^2]}_{=\mathrm{Var}[K]} + 2E[A \cdot B] + \underbrace{E[B^2]}_{=\mathrm{Var}[S]} \qquad (*)$$

K, S indep. $\Rightarrow A, B$ indep. r.v.'s

$$\Rightarrow E[A \cdot B] = E[A] \cdot E[B]$$

Recall:
X, Y indep. r.v.'s, i.e.
$\Pr(X \le x, Y \le y) = \Pr(X \le x) \cdot \Pr(Y \le y)$
$\Rightarrow E[X \cdot Y] = E[X]\, E[Y]$

$$\Rightarrow (*) = \mathrm{Var}[K] + 2 \underbrace{E[A]}_{=0} \cdot \underbrace{E[B]}_{=0} + \mathrm{Var}[S]$$

$$= \mathrm{Var}[K] + \mathrm{Var}[S] \qquad (**)$$

(3) Ex. 2

$$\mathrm{Var}[S] = \mathbb{E}[(S-\mathbb{E}[S])^2]$$

$g(u) \equiv 1$ density of $\Pr(S \le u)$

$$\Rightarrow \mathbb{E}[S] \overset{(3.1.5)}{=} \int_0^1 u \cdot 1 \, du = \frac{1}{2}$$

$$\Rightarrow \mathbb{E}[(S-\underbrace{\mathbb{E}[S]}_{=\frac{1}{2}})^2] = \int_0^1 (u-\tfrac{1}{2})^2 du$$

$$= \frac{1}{3}(u-\tfrac{1}{2})^3 \Big|_{u=0}^{1} = \frac{1}{12}$$

$$\Rightarrow (**) = \mathrm{Var}[K] + \frac{1}{12}$$

Problem 5:

age x	e_x
75	10.5
76	10.0
77	9.5

$$_2p_{75} \overset{\text{Prob. 1}}{=} p_{75} \cdot \underbrace{p_{75+1}}_{\text{cond. prob.}}$$

Hint $\rightarrow$ $p_{75} = \frac{e_{75}}{1+e_{75+1}} = \frac{10.5}{1+10.0}$

$$p_{75+1} = \frac{e_{75+1}}{1+e_{75+2}} = \frac{10.0}{1+9.5}$$

$$\Rightarrow {}_2p_{75} = 0.909$$

Remains to show: $e_{x+j+1} = \frac{e_{x+j}}{p_{x+j}} - 1$

W.l.o.g. $j=0$

$$\frac{e_x}{p_x} = \sum_{k\ge 1} \frac{{}_kp_x}{p_x} = \frac{\overbrace{{}_1p_x}^{=1}}{{}_1p_x} + \sum_{k\ge 2} \frac{{}_kp_x}{p_x}$$

$$= 1 + \underbrace{\sum_{l\ge 1} \frac{{}_{l+1}p_x}{p_x}}_{\overset{\text{def.}}{=} {}_lp_{x+1}} = 1 + e_{x+1}$$

$$\overset{\text{def}}{=} e_{x+1}$$

Ex. 2

Prob. 4: Consider an endowment; i.e. means a mixture of a term insurance and pure endowment; with PV:

$$Z = Z_1 + Z_2$$

where

$$Z_1 = v^{K+1} \mathbb{1}_{\{K<n\}}$$

PV of a term insur.

and

$$Z_2 = v^n \mathbb{1}_{\{K \geq n\}}$$

PV of a pure endowment

Observe that

$$Z_1 \cdot Z_2 = 0$$

$$\Rightarrow \quad \mathrm{Var}[Z] = E[(Z_1 - E[Z_1] + Z_2 - E[Z_2])^2]$$

$$= \mathrm{Var}[Z_1] + \mathrm{Var}[Z_2] + \underbrace{E[2 Z_1 Z_2]}_{=0} - 2 \underbrace{E[Z_1]}_{= A^1_{x:\overline{n}|}} \underbrace{E[Z_2]}_{= A_{x:\overline{n}|}^{\,1}}$$

$\mathrm{Var}[Z]$ risk measure; measures the risk of the financial position given by the pay off profile of Z

$\longrightarrow$ interpret:

$$\underbrace{\mathrm{Var}[Z_1] + \mathrm{Var}[Z_2]}_{\text{risk of selling the two insurances separately}} \geq \underbrace{\mathrm{Var}[Z]}_{\text{risk of selling these policies in one endowment}}$$

Prob. 7:

$$Z = v^{K+S^{(m)}}$$

PV of a payment of 1 at the end of the mth part of the remaining life time in the year of death

$$S^{(m)} = \frac{1}{m}[mS + 1]$$

e.g. $S = \frac{2}{m} + \frac{1}{2m} \longrightarrow S^{(m)} = \frac{3}{m}$

; rounds to the next upper multiple of $\frac{1}{m}$

Ass.: K, S indep, S unif. distr.

$$A_x^{(m)} \; ?$$

$$A_x^{(m)} = E[v^{K+S^{(m)}}] = E[v^{K+1} v^{-(1-S^{(m)})}]$$

$\overset{K, S^{(m)} \text{ indep.}}{=}$ $\underbrace{E[v^{K+1}]}_{= A_x} E[v^{-(1-S^{(m)})}]$ (*)

$$E[v^{-(1-S^{(m)})}] \overset{v = \frac{1}{1+i}}{=} \int_0^1 (1+i)^{1 - \frac{1}{m}[m \cdot u + 1]} \, du$$

$$= \sum_{\ell=1}^{m} \int_{[\frac{\ell-1}{m}, \frac{\ell}{m})} (1+i)^{1 - \underbrace{\frac{1}{m}[m \cdot u + 1]}_{= \frac{\ell}{m}}} \, du$$

(5.)

$$= \sum_{\ell=1}^{m} (1+i)\,(1+i)^{-\frac{\ell}{m}} \cdot \underbrace{\int_{(\frac{\ell-1}{m},\,\frac{\ell}{m}]} 1\,du}_{=\frac{1}{m}}$$

$$= (1+i)\underbrace{\left(\sum_{\ell=1}^{m} \frac{1}{m} v^{\frac{\ell}{m}}\right)}_{=a^{(m)}} \overset{\text{Ex. 2}}{=} (1+i)\,\frac{1-v}{i^{(m)}} = \underbrace{(1+i)(1-v)}_{=i} \cdot \frac{1}{i^{(m)}} = \frac{i}{i^{(m)}}$$

Prob. 8 :

$Z = C(T)\cdot v^T$

PV of a paym. of $C(T)$ at the instant of death

Ass.: $\mu_{x+j+u} = \mu_{x+j+\frac{1}{2}}$, $0 < u < 1$, $j \in \mathbb{N}_0$

$$E[Z] = \sum_{j\geq 0} \ell_{j+1}\, v^{j+1}\, {}_jp_x \cdot q_{x+j} \quad \text{with}$$

$$\ell_{j+1} = E\left[C(j+S)(1+i)^{1-S} \mid K=j\right] \qquad (*)$$

ℓ_{j+1} ?

$$\Pr(S\leq u \mid K=j) = \frac{\Pr(j\leq T < j+u)}{\Pr(K=j)} \overset{\text{def. } {}_ju q}{=} \frac{{}_ju q}{{}_jp_x \cdot q_{x+j}} \overset{\text{Ex. 2, Pr. 2}}{=} {}_jp_x \cdot {}_u q_{x+j}$$

$$= \frac{{}_u q_{x+j}}{q_{x+j}}$$

$${}_u p_{x+j} \overset{(3.2.2)}{=} e^{-\int_0^u \mu_{x+j+s}\,ds} = e^{-u\,\mu_{x+j+\frac{1}{2}}}$$

$$\Rightarrow \frac{d}{du}\Pr(S\leq u \mid K=j) = \frac{d}{du}\left(\frac{1-e^{-u\mu_{x+j+\frac{1}{2}}}}{q_{x+j}}\right)$$

$$= \mu_{x+j+\frac{1}{2}}\,\frac{e^{-u\mu_{x+j+\frac{1}{2}}}}{q_{x+j}} = \mu_{x+j+\frac{1}{2}}\,\frac{\left(\overbrace{e^{-\mu_{x+j+\frac{1}{2}}}}^{\text{ass. } p_{x+j}}\right)^u}{\underbrace{q_{x+j}}_{=1-p_{x+j}}}$$

$$= \mu_{x+j+\frac{1}{2}}\,\frac{(p_{x+j})^u}{1-p_{x+j}} \quad \text{cond. density}$$

$\overset{(*)}{\Rightarrow}$ cond. dens.

$$\ell_{j+1} = \int_0^1 C(j+u)(1+i)^{1-u}\left(\mu_{x+j+\frac{1}{2}}\,\frac{(p_{x+j})^u}{1-p_{x+j}}\right)du$$

$C(t) = e^{\lambda t} \Rightarrow C(j+u) = e^{\lambda(j+u)}$

$(1+i) = e^{\delta} \Rightarrow (1+i)^{1-u} = e^{\delta(1-u)}$

$(p_{x+j})^u = e^{u\log p_{x+j}}$

(6)

$$\Rightarrow l_{j+1} = \frac{\mu_{x+j+\frac{1}{2}}\, e^{\hat{r}\cdot j+\delta}}{1-p_{x+j}} \int_0^1 e^{u(\hat{r}-\delta+\log p_{x+j})}\, du$$

$$= \frac{\mu_{x+j+\frac{1}{2}} \cdot e^{\hat{r}\cdot j+\delta}}{1-p_{x+j}} \cdot \frac{e^{\hat{r}-\delta}\cdot p_{x+j} - 1}{\hat{r}-\delta+\log p_{x+j}}$$

$$= -\mu_{x+j+\frac{1}{2}}$$