Exercises 3

Prob. 1 : $Z = c(K)\, v^{K+1} \mid \{K<3\}$

$\longrightarrow$ Pv of a paym. of $c(K)$ at $t=K+1$, if $K<3$

$\Pi := E[Z]$

$Pr(Z > \Pi)$?

$$\Pi = E[Z] = \sum_{j=0}^{2} c(j)\, v^{j+1}\, {}_jP_x \cdot q_{x+j}$$

${}_1P_x = 1 - q_x = 1 - q_{x+0} = 0.8$

${}_2P_x = P_x \cdot \underbrace{P_{x+1}}_{1-q_{x+1}} = 0.8 \cdot 0.75$

$$\Rightarrow \quad \Pi = 3 \cdot 0.9 \cdot 1 \cdot 0.2 + 2 \cdot (0.9)^2 \cdot 0.8 \cdot 0.25 + 1 \cdot (0.9)^3 \cdot 0.8 \cdot (0.75) \cdot (0.5) = 1.0827$$

$$\{Z > \Pi\} = \{K=0\} \cup \{K=1\}$$

$$\Rightarrow \quad Pr(Z > \Pi) = \underbrace{Pr(K=0)}_{=q_x} + \underbrace{Pr(K=1)}_{P_x \cdot q_{x+1}} = 0.4$$

Prob. 2 :

$$\frac{d}{dx}\overline{A}_x - (\delta + \mu_x)\overline{A}_x = -\mu_x$$

prod. rule $\Longleftrightarrow$ $\int_0^h (\dots)\, dx$

$$\frac{d}{dx}\left(e^{-\int_0^x (\delta+\mu_s)\, ds}\, \overline{A}_x\right) = -\mu_x\, e^{-\int_0^x (\delta+\mu_s)\, ds}$$

$$\Longrightarrow \quad e^{-\int_0^h (\delta+\mu_s)\, ds}\, \overline{A}_h - \overline{A}_0 = -\int_0^h \mu_x\, e^{-\int_0^x (\delta+\mu_s)\, ds}\, dx$$

$$\Longrightarrow \quad \overline{A}_h = e^{\int_0^h (\delta+\mu_s)\, ds}\, \overline{A}_0 - \int_0^h \mu_x\, e^{\int_x^h (\delta+\mu_s)\, ds}\, dx$$

Prob. 3: whole life annuity-due

$\longrightarrow$ PV $Y = \sum_{j=0}^{K} v^j$

$\longrightarrow$ net sing. prem.:

$$\ddot{a}_x := E[Y] = \sum_{j\geq 0} v^j \cdot {}_jp_x$$

$$\Rightarrow \ddot{a}_{x+1} = \sum_{j\geq 0} v^j \cdot {}_jp_{x+1} \qquad (*)$$

Recall:

$${}_{j-1}p_{x+1} \overset{\text{def}}{=} \frac{{}_jp_x}{p_x} \iff {}_jp_x = p_x \cdot {}_{j-1}p_{x+1} \qquad (**)$$

$$\Rightarrow \ddot{a}_x = \sum_{j\geq 0} v^j {}_jp_x = 1 + \sum_{j\geq 1} v^j \cdot {}_jp_x$$

$$\overset{(**)}{=} 1 + \sum_{j\geq 1} v^j \cdot p_x \cdot {}_{j-1}p_{x+1} = 1 + \sum_{\ell\geq 0} v^{\ell+1} \cdot p_x \cdot {}_\ell p_{x+1}$$

$$= 1 + v \cdot p_x \underbrace{\left(\sum_{\ell\geq 0} v^\ell \cdot {}_\ell p_{x+1} \right)}_{\overset{(*)}{=} \ddot{a}_{x+1}} \qquad (***)$$

p_{73} ?

Use $(***)$, that is

$$\ddot{a}_x = 1 + v \cdot \ddot{a}_{x+1} \cdot p_x$$

$$\Rightarrow \ddot{a}_{73} = 1 + v \cdot \ddot{a}_{74} \cdot p_{73}$$

$v = \frac{1}{1+0.03}$, $\ddot{a}_{73} = 7.73$, $\ddot{a}_{74} = 7.43$

$$\Rightarrow p_{73} = \frac{6.73}{7.43} \cdot 1.03 \approx 0.93$$

Prob. 4: Def.

$$PV_{j+1} := \sum_{\ell=0}^{j} (2+\ell) v^\ell \;, \; j = 0, 1, 2$$

$\longrightarrow$ PV of the 3 year temp. life ann.-due

$$Y = PV_{K+1} \, 1_{\{K<3\}} + PV_3 \, 1_{\{K\geq 3\}}$$

Var[Y] ?

(3) Ex. 3

$$\mathrm{Var}[Y] = E[(Y - E[Y])^2]$$
$$= E[Y^2] - (E[Y])^2$$

1. $$E[Y] = \sum_{j=0}^{2} PV_{j+1} \cdot \underbrace{Pr(K=j)}_{{}_jp_x \cdot \underbrace{q_{x+j}}_{(1-p_{x+j})}} + PV_3 \cdot \underbrace{Pr(K \geq 3)}_{={}_3p_x}$$
$$= \sum_{j=0}^{2} PV_{j+1} \cdot {}_jp_x (1 - p_{x+j}) + PV_3 \cdot {}_3p_x$$

$$\left.\begin{aligned} {}_2p_x &= p_x \cdot p_{x+1} \\ {}_3p_x &= \underbrace{p_x \cdot p_{x+1}}_{={}_2p_x} p_{x+2} \end{aligned}\right\} \text{ Exerc. 2, Prob. 1}$$

$$\Rightarrow \quad {}_2p_x = 0.8 \cdot 0.75 = 0.6$$
$$ {}_3p_x = 0.6 \cdot 0.5 = 0.3$$

$$PV_1 = 2 \quad , \quad PV_2 = 2 + 3 \cdot \underbrace{0.9}_{=v} = 4.7$$
$$PV_3 = 2 + 3 \cdot 0.9 + 4 \cdot (0.9)^2 = 7.94$$

$$\Rightarrow \quad E[Y] = 2 \cdot 1 \cdot (1-0.8) + 4.7 \cdot 0.8 \, (1-0.75) + 7.94 \cdot 0.3 + 7.94 \cdot 0.6 \cdot (1-0.5) = 6.104$$

2. $$E[Y^2] = \sum_{j=0}^{2} (PV_{j+1})^2 \, {}_jp_x (1 - p_{x+j}) + (PV_3)^2 \, {}_3p_x$$
$$= 2^2 \cdot 1 \cdot (1-0.8) + (4.7)^2 \cdot 0.8 \, (1-0.75) + (7.94)^2 \cdot 0.6 \, (1-0.5) + (7.94)^2 \cdot 0.3$$
$$= 43.04416$$

$$\Rightarrow \mathrm{Var}[Y] \approx 5.785$$

Solution to Exerc. 3, Prob. 5

Problem 5 (i) We know that

$$ {}_tp_x = \exp(-\int_0^t \mu_{x+s} ds). $$

So

$$ {}_{s_i-t_i}p_{x+t_i} = \exp(-\int_0^{s_i-t_i} \mu_{x+t_i+s} ds) = \exp(-\int_{t_i}^{s_i} \mu_{x+s} ds). $$

hence we get

$$ L = (\mu_{x+\frac{1}{2}})^{D_x} \exp(-\mu_{x+\frac{1}{2}} E_x). $$

Then maximization of L with respect to $\mu_{x+\frac{1}{2}}$ gives

$$ \widehat{\mu}_{x+\frac{1}{2}} = \frac{D_x}{E_x} $$

(ii) Suppose for example that one of the two lives $(x+0.4)$ dies at $x+0.6$. This implies that the other live of the two either leaves at $x+0.5$, $x+0.8$ or $x+1$. So the exposure must be

$$ \begin{aligned} E_x &= ((x+0.6)-(x+0.4)) + ((x+0.5)-(x+0.4)) + ((x+0.2)-x) + 2 \times ((x+0.8)-x)) \\ &\quad +7 \times ((x+1)-x)) \\ &= 9.1. \end{aligned} $$

All the other possibilities must also give the same value, that is

$$ E_x = 9.1. $$

Since

$$ \widehat{\mu}_{x+\frac{1}{2}} = \frac{D_x}{E_x} $$

and $D_x = 1$ we see that

$$ \widehat{\mu}_{x+\frac{1}{2}} = \frac{1}{9.1} = 0.1098901. $$