Exercises 4

Problem 1:

$Z = v^{K+1}$

→ PV of a paym. 1 at $t = K+1$

$V = G \cdot \ddot{a}_{\overline{K+1}|}$ ← whole life annuity-due

→ PV of annu. prem. G from $t = 0, \dots, K$

Consider

$L = Z - V$, if $G = P_x$

and

$L^* = Z - V$ with G s.t.

$E[L^*] = -0.20$

$$\ddot{a}_{\overline{K+1}|} = \frac{1 - v^{K+1}}{d} \qquad (1)$$

$$\Rightarrow L = \left(1 + \frac{P_x}{d}\right) v^{K+1} - \frac{P_x}{d} \Rightarrow$$

$$\underbrace{Var[L]}_{= 0.30} = \left(1 + \frac{P_x}{d}\right)^2 Var[v^{K+1}] \qquad (2)$$

Similarly

$$Var[L^*] = \left(1 + \frac{G}{d}\right)^2 Var[v^{K+1}] \qquad (3)$$

On the other hand

$$\underbrace{E[L]}_{=0} = A_x - P_x \ddot{a}_x \overset{(1)}{=} (1 - d\ddot{a}_x) - P_x \ddot{a}_x$$

$$= 1 - \left(1 + \frac{P_x}{d}\right) d \ddot{a}_x \qquad (4)$$

Similarly

$$\underbrace{E[L^*]}_{= -0.2} = 1 - \left(1 + \frac{G}{d}\right) d \ddot{a}_x \qquad (5)$$

(4), (5) $\Rightarrow$

$$\frac{\left(1 + \frac{G}{d}\right)}{\left(1 + \frac{P_x}{d}\right)} = \frac{1 + 0.2}{1} = 1.2$$

Further (2), (3) give

$$\frac{\underbrace{Var[L^*]}_{= 0.30}}{Var[L]} = (1.2)^2 \Rightarrow Var[L^*] = 1.44 \cdot 0.3 =$$

0.432

(2)

Problem 2: Q ann. prem.

$$Z = \left(\frac{Q}{2} + 10.000\right) v^{K+1} \mid_{\{K<20\}} + 10.000\, v^{K+1} \mid_{\{K\geq 20\}}$$

→ PV of a paym. of $\left(\frac{Q}{2} + 10.000\right)$ at $t = K+1$, if $K < 20$
+ PV of a paym. of 10.000 at $t = K+1$, if $K \geq 20$

$$V = Q\, \ddot{a}_{\overline{K+1}} \mid_{\{K<20\}} + Q\, \ddot{a}_{\overline{20}} \mid_{\{K\geq 20\}}$$

→ PV of ann. prem. Q until $t = \min(19, K)$

$$Z = \frac{Q}{2} v^{K+1} \mid_{\{K<20\}} + 10.000\, v^{K+1}$$

$$v^n = 1 - d\, \ddot{a}_{\overline{n}} \quad (*)$$

$$\Rightarrow Z = \frac{Q}{2} \mid_{\{K<20\}} + 10.000\, v^{K+1} - \frac{Q}{2} \cdot d\, \ddot{a}_{\overline{K+1}} \mid_{\{K<20\}}$$

$$\Rightarrow L = Z - V = \frac{Q}{2} \mid_{\{K<20\}} + 10.000\, v^{K+1} - Q\left(1 + \frac{d}{2}\right) \ddot{a}_{\overline{K+1}} \mid_{\{K<20\}} - Q\, \ddot{a}_{\overline{20}} \mid_{\{K\geq 20\}}$$

$$\overset{(*)}{=} \frac{Q}{2}\left(1 - v^{20} \mid_{\{K\geq 20\}}\right) + 10.000\, v^{K+1} - Q\left(1+\frac{d}{2}\right)\left(\ddot{a}_{\overline{K+1}} \mid_{\{K<20\}} + \ddot{a}_{\overline{20}} \mid_{\{K\geq 20\}}\right)$$

$E[\cdot]$ ⇒ $0 = E[L] = \frac{Q}{2}\left(1 - v^{20}\, {}_{20}p_x\right) + 10.000\, A_x - Q\left(1+\frac{d}{2}\right) \ddot{a}_{x:\overline{20}}$

Problem 3: 9-year immediate life ann.

→ $Y = a_{\overline{K}} \mid_{\{K<9\}} + a_{\overline{9}} \mid_{\{K\geq 9\}}$

→ PV of an ann. paym of 1 from $t = 1, \dots, \min(9, K$

⇒ $a_{30:\overline{9}} := E[Y] = \ddot{a}_{30:\overline{10}} - 1$

$$A_{30:\overline{10}} = \underbrace{E\left[v^{K+1} \mid_{\{K<10\}}\right]}_{= A^{1}_{30:\overline{10}} \text{ term insurance}} + \underbrace{E\left[v^{n} \mid_{\{K\geq 10\}}\right]}_{= A^{1}_{30:\overline{10}} \text{ pure endowment}}$$

$$= 1 - d\, \ddot{a}_{30:\overline{10}}$$

since $\ddot{a}_{\overline{n}} = \frac{1 - v^n}{d}$

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(3) $d = \frac{i}{1+i}$

$\Rightarrow \quad A_{30:\overline{10}|} = 0.4$

$\overset{(*)}{\Rightarrow} \quad A^1_{30:\overline{10}|} = A_{30:\overline{10}|} - v^{10}\,{}_{10}p_{30} = 0.4 - 0.035 = 0.05$

$\Rightarrow \quad 1000\, P^1_{30:\overline{10}|} = 1000\, \frac{A^1_{30:\overline{10}|}}{\ddot{a}_{30:\overline{10}|}} = 7.58$

Problem 4 : Recall that

(4) ${}_jL$ = PV of the future benefit paym. at time $t=j$
− PV of the future premium paym. at time $t=j$

${}_0L = L$

${}_jV$ net premium reserve def. by

$${}_jV = E[\,{}_jL \mid T > j\,]$$

To make the ins. policy interesting for the insured and the insurer
→ reasonable to require that

1. ${}_0V = E[L] = 0$ (equiv. principle)
2. ${}_jV \geq 0$ for all j

3-year endowment with paym. C_{K+1} ← endowment amount

→ PV $Z = C_{K+1} v^{K+1} \mathbb{1}_{\{K<3\}} + {}_3V \cdot v^3 \mathbb{1}_{\{K\geq 3\}}$

recursion formula:

$$({}_jV + \pi_j)(1+i) = E[\,C_{j+1} \cdot q_{x+j} + {}_{j+1}V \cdot p_{x+j}\,]$$

for $\pi_0 = \pi_1 = \pi_2 = 1$, $\pi_3 = -{}_3V$, $\pi_4 = \pi_5 = \ldots = 0$

and $C_1 = 2$, $C_2 = 3$, $C_3 = 4$, $C_4 = C_5 = \ldots = 0$ (see Rem. 7.2)

$$\underbrace{{}_0V}_{0} + \pi_0 = 0.9 \cdot [\,2 \cdot 0.20 + {}_1V \cdot (1 - 0.20)\,]$$

$$\Rightarrow {}_1V = 0.8889$$

$$\Rightarrow {}_2V = 1.7984$$

$$\Rightarrow {}_3V = 2.2187$$

(5) Ex. 4

Problem 5: $2V_{x:\overline{4}|}$?

PV of 4-year endowment:

$$Z = v^{K+1} \mathbb{1}_{\{K<4\}} + v^4 \mathbb{1}_{\{K\geq 4\}}$$

→ PV of a term ins. + PV of a pure endowment

PV of premiums:

$$V = \Pi \cdot \ddot{a}_{\overline{K+1}|} \mathbb{1}_{\{K<4\}} + \Pi \ddot{a}_{\overline{4}|} \mathbb{1}_{\{K\geq 4\}}$$

→ PV of annual paym. of the prem. Π from $t = 0, \dots, \min(3, K)$

⇒ V PV of 4-year temp. life annuity-due

An endowment is a special case of a general type of life insurance with

$$L = c_{K+1} v^{K+1} - \sum_{j=0}^{K} \Pi_j \cdot v^j$$

for $c_1, c_2, \dots, c_n = 1$, $c_{n+1} = c_{n+2} = \dots = 0$

and

$$\Pi_0 = \Pi_1 = \dots = \Pi_{n-1} = P_{x:\overline{n}|}, \ \Pi_n = -1, \ \Pi_{n+1} = \Pi_{n+2} = \dots = 0$$

→ recursion:

$$_kV + \Pi_k = c_{k+1} v + (_{k+1}V - c_{k+1}) v \, p_{x+k} \qquad (*)$$

See Section 6 in Gerber.

$$P_{x:\overline{4}|} = \frac{A_{x:\overline{4}|}}{\ddot{a}_{x:\overline{4}|}} = E[L_V]$$

(← net single prem. of the endowm.)

$$\ddot{a}_{x:\overline{4}|} = \sum_{j=0}^{3} \ddot{a}_{\overline{j+1}|} \, {}_j p_x \cdot q_{x+j} + \ddot{a}_{\overline{4}|} \, {}_4 p_x \qquad (**)$$

On the other hand

$$\ddot{a}_{x:\overline{4}|} = \frac{1 - A_{x:\overline{4}|}}{d} \qquad (***)$$

$$d = \frac{i}{1+i}$$

$$\ddot{a}_{\overline{2}|} = 1 + v = 1.930 \quad \overset{v = \frac{1}{1+i}}{\Longrightarrow} \quad i \approx 0.0753$$

$$\Rightarrow d \approx 0.070$$

Recall $\quad {}_{k-1|}q_x := Pr(k-1 \leq T \leq k) = {}_{k-1}p_x \cdot q_{x+k-1}$

$$\Rightarrow \sum_{\ell=1}^{K} {}_{\ell-1|}q_x = {}_K q_x \quad \Longrightarrow$$

(6) $q_{x+k-1} = \frac{{}_{k-1|}q_x}{{}_{k-1}p_x}$

with ${}_kp_x = 1 - \sum_{\ell=1}^{k} {}_{\ell-1|}q_x$

$\Rightarrow$ $p_x = 0.67$, ${}_2p_x = 0.43$, ${}_3p_x = 0.27$, ${}_4p_x = 0.16$

$q_x = 0.33$, $q_{x+1} \approx 0.358$, $q_{x+2} \approx 0.372$, $q_{x+3} \approx 0.4074$

(**) $\Rightarrow$ $\ddot{a}_{x:\overline{4}|} \approx 2.2121$

(***) $\Rightarrow$ $A_{x:\overline{4}|} \approx 0.845$ $\Rightarrow$ $P_{x:\overline{4}|} \approx 0.382$

(*) $\Rightarrow$ ${}_1V \approx 0.1205$ $\overset{(*)}{\Rightarrow}$ ${}_2V = {}_2V_{x:\overline{4}|} \approx 0.284$

(7) Problem 6:

Recall: The total loss L_j of the insurer during year $j+1$ is given by

$$L_j = \underbrace{(C_{j+1} v - ({}_jV + \Pi_j))}_{\text{PV at } t=j \text{ of the paym. } C_{j+1} \text{ at } j+1} 1_{\{K=j\}} + \underbrace{({}_{j+1}V \cdot v - ({}_jV + \Pi_j))}_{\text{PV at } t=j \text{ of the reserve } {}_{j+1}V \text{ at } j+1} \cdot 1_{\{K \geq j+1\}}$$

Note that

$$E[L_j] = 0 \qquad \text{(Section 6.7 in Gerber)}$$

$$\Rightarrow \operatorname{Var}[L_1] = E[L_1^2]$$

$$= (\underbrace{C_2}_{=3} v - ({}_1V + \Pi_1))^2 \underbrace{Pr(K=1)}_{=1p_x \cdot q_{x+1} = 0.8 \cdot 0.25}$$

$$+ ({}_2V \cdot v - ({}_1V + \Pi_1))^2 \underbrace{Pr(K \geq j+1)}_{={}_2p_x = p_x \cdot p_{x+1}}$$

with ${}_2V = 1.7984$, $v = 0.9$, ${}_1V = 0.8889$, $\Pi_1 = 1$, $p_x = 0.8$, $p_{x+1} = 1 - q_{x+1} = 0.75$

$$\Rightarrow \operatorname{Var}[L_1] \approx 0.1754$$