Exercises 6

Prob. 1: Recall from Ex. 5, Prob. 1 that

$\hat{\lambda} = \frac{n}{T_n}$ ← sample size is the maximum likelihood estimator for the intensity

SLLN ⟶ $\frac{1}{\hat{\lambda}} = \frac{1}{n} T_n = \frac{1}{n} \sum_{i=1}^{n} W_i \xrightarrow[n\to\infty]{} E[W_1]$

$\Rightarrow \hat{\lambda} \approx 1/$ sample mean of W_i in a certain period

intensities $\hat{\lambda}$ vary over time

⟶ reasonable: $N(t)$ modelled by an inhom. Poisson proc. with

$$\mu(t) = (t-(i-1))\hat{\lambda}_{1980+i-1} + \sum_{j=1}^{i-1} \hat{\lambda}_{1980+j-1}, \text{ if } i-1 \le t < i, \; i = 1, \dots, 11$$ (t

⟶ $E[\hat{S}(t)]$?

where $\hat{S}(t) = \sum_{i=1}^{N(t)} e^{-r T_i} X_i$ ← discount factor

$r = 0.03$, $t = 10$ years, $E[X_1] = 2.2$

We know that

$$N(t) \overset{d}{=} \hat{N}(\mu(t))$$

and

$$T_i \overset{d}{=} \mu^{-1}(\hat{T}_i)$$

for the arrival times $\hat{T}_i$ of a hom. Poisson proc $\hat{N}$ with $\lambda = 1$

$$\Rightarrow \hat{S}(t) = \sum_{i=1}^{\hat{N}(\mu(t))} e^{-r \cdot \mu^{-1}(\hat{T}_i)} \cdot X_i$$

$$= \sum_{i=1}^{\hat{N}(\mu(t))} g(\hat{T}_i, X_i)$$

Prop. 2.1.16 in Mikosch $\Rightarrow$ $\hat{S}(t) \overset{d}{=}$ ← distribution $\sum_{i=1}^{\hat{N}(\mu(t))} g(\mu(t)\cdot U_i, X_i)$

$$= \sum_{i=1}^{\hat{N}(\mu(t))} e^{-r \cdot \mu^{-1}(\mu(t)\cdot U_i)} \cdot X_i$$

for i.i.d $U_i \sim U(0,1)$ indep. of $X_i, \hat{T}_i$

$$\Rightarrow E[\hat{S}(t)] = E\left[\sum_{i=1}^{\hat{N}(\mu(t))} e^{-r \cdot \mu^{-1}(\mu(t)\cdot U_i)} \cdot X_i\right]$$

(2)

$$= E\Big[\sum_{n\geq 0}\Big(\sum_{i=1}^{n} e^{-\tau\cdot\mu^{-1}(\mu(t)U_i)}X_i^2\Big)\cdot 1_{\{\hat N(\mu(t))=n\}}\Big]$$

$$= \sum_{n\geq 0} E\Big[\Big(\sum_{i=1}^{n} e^{-\tau\cdot\mu^{-1}(\mu(t)U_i)}X_i^2\Big)\cdot 1_{\{\hat N(\mu(t))=n\}}\Big]$$

($\hat N(\mu(t))$ and (U_i, X_i^2) indep. r.v.'s)

$$= \sum_{n\geq 0} E\Big[\sum_{i=1}^{n} e^{-\tau\cdot\mu^{-1}(\mu(t)U_i)}X_i^2\Big]\cdot E\big[1_{\{\hat N(\mu(t))=n\}}\big]$$

($e^{-\tau\cdot\mu^{-1}(\mu(t)U_i)}$ and X_i^2 indep. r.v.'s; $E[1_{\{\hat N(\mu(t))=n\}}] = P(\hat N(\mu(t))=n)$)

$$= \sum_{n\geq 0} P(\hat N(\mu(t))=n)\sum_{i=1}^{n} E\big[e^{-\tau\cdot\mu^{-1}(\mu(t)U_i)}\big]\,E[X_i^2]$$

(U_i, X_i^2 i.i.d.)

$$= \Big(\sum_{n\geq 0} n\cdot P(\hat N(\mu(t))=n)\Big)\,E\big[e^{-\tau\cdot\mu^{-1}(\mu(t)U_1)}\big]\cdot E[X_1^2] \quad (*)$$

($\sum_{n\geq 0} n\cdot P(\hat N(\mu(t))=n) = E[\hat N(\mu(t))] = \mu(t)$; $E[e^{-\tau\cdot\mu^{-1}(\mu(t)U_1)}] =: J$)

$t=10$ $\overset{(*)}{\Rightarrow}$ $\mu(10) = 5.34$ (×365 days)

$$J = \int_0^1 e^{-0.03\cdot\mu^{-1}(5.34\cdot x)}\,dx \overset{x=\frac{\mu(u)}{5.34}}{=} \frac{1}{5.34}\int_0^{10}\sum_{i=1}^{11} i\cdot 1_{[i-1,i)}(u)\cdot\mu'(u)\,e^{-0.03\cdot u}\,du$$

($\mu'(u) = \lambda_{1980+i-1}$ on $[i-1,i)$)

$$= \frac{1}{5.34}\sum_{i=1}^{10} i\cdot\lambda_{1980+i-1}\Big(-\frac{1}{0.03}e^{-0.03\cdot u}\Big|_{u=i-1}^{i}\Big)$$

$$\approx 0.85459$$

$\overset{(*)}{\Rightarrow}$ $E[\hat S(t)] \approx 5.34\cdot 365\cdot 0.85459\cdot 2.2 \approx 3664.5$

Prob. 2 : $E[\hat N(t)]$?

where $\hat N(t) = \sum_{i=1}^{N(t)} 1_{[0,t]}(T_i+V_i)$

$\longrightarrow$ # of claims reported up to time t

V_i delays i.i.d $U(0,1)$ indep. of (T_i)

$t=1$

Regard V_i as claim X_i^2

Prop. 2.1.16 in Mikosch $\Rightarrow$ $\tilde N(t) \overset{d}{=} \sum_{i=1}^{N(t)} 1_{[0,t]}(t\cdot U_i+V_i)$

with i.i.d $U_i\sim U(0,1)$ indep. of (T_i), (V_i)

(3)

$$\Rightarrow \quad E[\hat{N}(t)] = E\Big[\sum_{i=1}^{N(t)} 1_{[0,t]}(t\cdot U_i + V_i)\Big]$$

$$= E\Big[\sum_{n\geq 0}\Big(\sum_{i=1}^{N(t)} 1_{[0,t]}(t\cdot U_i + V_i)\Big)\cdot 1_{\{N(t)=n\}}\Big]$$

(indep.)

$$= \sum_{n\geq 0} \underbrace{E\Big[\sum_{i=1}^{n} 1_{[0,t]}(t\cdot U_i + V_i)\Big]}_{= n\cdot E[1_{[0,t]}(t\cdot U_1 + V_1)]} \cdot P(N(t)=n)$$

$$= \underbrace{E[N(t)]}_{=t} \cdot \underbrace{E[1_{[0,t]}(t\cdot U_1 + V_1)]}_{=:\,J}$$

$$J = \int_0^1\int_0^1 1_{[0,t]}(t\cdot x_1 + x_2)\,dx_1\,dx_2$$

$$= \int_0^{1\wedge t} \frac{1}{t}(t - x_2)\,dx_2 = (1\wedge t) - \frac{1}{t}\frac{(1\wedge t)^2}{2}$$

$$\Rightarrow E[\hat{N}(t)] = t\cdot\left((1\wedge t) - \frac{1}{t}\frac{(1\wedge t)^2}{2}\right)$$

$$\Rightarrow E[\hat{N}(t)] \approx t = E[N(t)]$$

for large t

Exerc. 6

Prob. 3 : $P(T_1 \le x_1, T_2 \le x_2) = P(\overbrace{N(x_1)}^{=N(x_2)-N(x_1)+N(x_1)} \ge 1, N(x_2) \ge 2)$

$$= E\Big[P(N(x_1) \ge 1, N(x_2)-N(x_1)+N(x_1) \ge 2 \mid N(x_1)) \Big]$$

$$= E\Big[\sum_{j \ge 0} P(N(x_1) \ge 1, N(x_2)-N(x_1)+N(x_1) \ge 2 \mid N(x_1) = j) \cdot \mathbb{1}_{\{N(x_1)=j\}} \Big]$$

$$= E\Big[\sum_{j \ge 0} P(j \ge 1, N(x_2)-N(x_1)+j \ge 2 \mid N(x_1) = j) \cdot \mathbb{1}_{\{N(x_1)=j\}} \Big]$$

$$= \sum_{j \ge 0} P(j \ge 1, N(x_2)-N(x_1)+j \ge 2 \mid N(x_1)=j) \underbrace{E[\mathbb{1}_{\{N(x_1)=j\}}]}_{=P(N(x_1)=j)}$$

$$= P(N(x_2)-N(x_1)+1 \ge 2) P(N(x_1)=1)$$

$$+ \sum_{j \ge 2} \underbrace{P(j \ge 1, \overbrace{N(x_2)-N(x_1)}^{\ge 0}+j \ge 2)}_{=1 \text{ for } j \ge 2} P(N(x_1)=j)$$

$$= \underbrace{P(N(x_2)-N(x_1) \ge 1)}_{=1-P(N(x_2)-N(x_1)=0)} P(N(x_1)=1) + \underbrace{\sum_{j \ge 2} P(N(x_1)=j)}_{= P(N(x_1) \ge 2) = 1-P(N(x_1) \le 1) = 1-P(N(x_1)=0)-P(N(x_1)=1)}$$

Poisson distr. $\big(1-e^{-\lambda(x_2-x_1)}\big) e^{-\lambda x_1} \lambda x_1 + 1 - e^{-\lambda x_1} - e^{-\lambda x_1} \lambda x_1$

$$= 1 - e^{-\lambda x_2} \lambda x_1 - e^{-\lambda x_1}$$

On the other hand,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(w_1,w_2) g(w_1,w_2) dw_1 dw_2 = \int_0^{x_1} \int_0^{x_2-w_2} \lambda e^{-\lambda w_1} \lambda e^{-\lambda w_2} dw_1 dw_2$$

$$= 1 - e^{-\lambda x_2} \lambda x_1 - e^{-\lambda x_1}.$$