STK 3505

Exercises 8

Prob. 1: $N(t)$ inhom. Poisson proc. with mean value function

$$\mu(t) = \int_0^t \lambda(s)\, ds,$$

where

$$\lambda(t) = \sum_{i=1}^{11} \hat{\lambda}_{1980+i-1} \mathbb{1}_{[i-1,\, i)}(t)$$

$$\Rightarrow \quad \mu(t) = (t-i+1)\hat{\lambda}_{1980+i-1} + \sum_{j=1}^{i-1} \hat{\lambda}_{1980+j-1} \qquad (+)$$

if $i-1 \le t < i$, $i = 1, \dots, 11$

1. $E[S(t)]$: $N(t) \overset{d}{=} \hat{N}(\mu(t))$, $\hat{N}$ standard Poisson proc.

We know from Prob. 1 of Exerc. 6 that

$$E\Big[\sum_{i=1}^{\hat{N}(t)} X_i\Big] = E[\hat{N}(t)] \cdot E[X_1]$$

$$\Rightarrow \quad E[S(t)] = E\Big[\sum_{i=1}^{\hat{N}(\mu(t))} X_i\Big] = \mu(t) \cdot E[X_1]$$

($\hat{N}(t) = t$)

$\mu(9) = 4.7$ because of $(+)$

$$\Rightarrow \quad E[S(t)] = 4.7 \cdot 365 \cdot 2.2 = 3774.1 \text{ in millions of Danish Krone}$$

2. $\mathrm{Var}[S(t)]$: $\mathrm{Var}[S(t)] = E[(S(t))^2] - (E[S(t)])^2 \qquad (++)$

($(E[S(t)])^2 = (\mu(t))^2 (E[X_1])^2$)

$$E[(S(t))^2] = E\Big[\Big(\sum_{i=1}^{N(t)} X_i\Big)^2\Big]$$

$$= E\Big[\sum_{n\ge 0} \Big(\sum_{i=1}^{n} X_i\Big)^2 \cdot \mathbb{1}_{\{N(t)=n\}}\Big] = \sum_{n\ge 0} E\Big[\Big(\sum_{i=1}^{n} X_i\Big)^2 \cdot \mathbb{1}_{\{N(t)=n\}}\Big]$$

(X_i indep. of $\mathbb{1}_{\{N(t)=n\}}$)

$$= \sum_{n\ge 0} \underbrace{E\Big[\Big(\sum_{i=1}^{n} X_i\Big)^2\Big]}_{\overset{X_i \text{ i.i.d}}{=} n\,\mathrm{Var}[X_1] + n^2 (E[X_1])^2} \cdot P(N(t)=n)$$

$$= \Big(\underbrace{\sum_{n\ge 0} n\, P(N(t)=n)}_{= E[N(t)] = \mu(t)}\Big) \mathrm{Var}[X_1] + \Big(\underbrace{\sum_{n\ge 0} n^2 P(N(t)=n)}_{= E[(N(t))^2] = (\mu(t))^2 + \mu(t)}\Big) (E[X_1])^2$$

Recall: $X \sim \mathrm{Pois}(\lambda) \Rightarrow E[X^2] = \lambda^2 + \lambda$

$$\Rightarrow \quad E[(S(t))^2] = \mu(t)\,\mathrm{Var}[X_1] + \big((\mu(t))^2 + \mu(t)\big)(E[X_1])^2$$

(2)

$\overset{(++)}{\Longrightarrow}$ $Var[S(t)] = \mu(t)\left(Var[X_1] + (E[X_1])^2\right)$

$= 4.7 \cdot (1.63 + (2.2)^2) = 30.409 \quad (\times 365 \text{ days})$

(3)

Prob. 2 : $N(t) = \hat{N}(\theta \mu(t))$ mixed Poisson proc.

$\theta > 0$ mixing variable (non-constant)

$\hat{N}$ standard Poisson proc. (i.e $\lambda = 1$)

$\mu(t)$ mean value function

$\theta, \hat{N}$ independent

Verify that

$$\mathrm{Var}[N(t)] > E[N(t)]$$

for any $t > 0$ with $\mu(t) > 0$

$\longrightarrow$ over-dispersion

$$\mathrm{Var}[N(t)] = E[(N(t))^2] - (E[N(t)])^2$$

1. $E[N(t)]$:

$$E[N(t)] = E[\hat{N}(\theta \mu(t))] = E\Big[\sum_{n \geq 0} \hat{N}(n \cdot \mu(t)) \cdot 1_{\{\theta = n\}}\Big]$$

$$= \sum_{n \geq 0} E[\hat{N}(n \cdot \mu(t)) \cdot 1_{\{\theta = n\}}]$$

($\hat{N}$ and $1_{\{\theta=n\}}$ indep.)

$$= \sum_{n \geq 0} \underbrace{E[\hat{N}(n \cdot \mu(t))]}_{n \cdot \mu(t)} P(\theta = n) = \sum_{n \geq 0} n \cdot P(\theta = n) \cdot \mu(t)$$

$$= \mu(t) E[\theta]$$

2. $E[(N(t))^2]$: Same procedure as in 1. :

$$E[(N(t))^2] = E[(\hat{N}(\theta \cdot \mu(t)))^2] = E\Big[\sum_{n \geq 0} (\hat{N}(n \cdot \mu(t)))^2 \cdot 1_{\{\theta = n\}}\Big]$$

$$= \sum_{n \geq 0} E[(\hat{N}(n \cdot \mu(t)))^2 \cdot 1_{\{\theta = n\}}] = \sum_{n \geq 0} \underbrace{E[(\hat{N}(n \cdot \mu(t)))^2]}_{(\mu(t))^2 n^2 + \mu(t) n} \cdot P(\theta = n)$$

($\hat{N}(n \cdot \mu(t))$ and $1_{\{\theta=n\}}$ indep.)

Recall : $X \sim \mathrm{Pois}(\lambda) \Rightarrow E[X^2] = \lambda^2 + \lambda$

$$\Rightarrow E[(N(t))^2] = (\mu(t))^2 \underbrace{\sum_{n \geq 0} n^2 P(\theta = n)}_{= E[\theta^2]} + \mu(t) \underbrace{\sum_{n \geq 0} n \cdot P(\theta = n)}_{= E[\theta]}$$

1.+2. $\Rightarrow$

$$\mathrm{Var}[N(t)] = \underbrace{(\mu(t))^2}_{>0} \underbrace{(E[\theta^2] - (E[\theta])^2)}_{>0 \text{ since } \mathrm{Var}[\theta] > 0 \text{ non-constant}} + \underbrace{\mu(t) E[\theta]}_{= E[N(t)]}$$

$$\Rightarrow \mathrm{Var}[N(t)] > E[N(t)]$$

Problem 3 We have real-life fire insurance data collected at Copenhagen Reinsurance in the period from 09/01/1990 until 10/01/1990, that is fire losses

$$X_1 = 14.851485, ..., X_8 = 5.726072$$

and inter-arrival times

$$W_1 = 7, ..., W_8 = 1.$$

(i) $p_{Net}(t)$ and $p_{EV}(t)$? for large t in the renewal model:

$$p_{Net}(t) := E[S(t)] \overset{\text{Prop. 3.1.3, Mikosch}}{\approx} t \cdot \frac{E[X_1]}{E[W_1]}$$

and

$$p_{EV}(t) := (1+\rho)E[S(t)] \overset{\text{Prop. 3.1.3, Mikosch}}{\approx} t \cdot (1+\rho) \cdot \frac{E[X_1]}{E[W_1]}$$

for large t, where $\rho = 0.04$.

We use the sample mean $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ to estimate $E[X_1]$ and $E[W_1]$ and get

$$E[X_1] \approx 4.187 \text{ and } E[W_1] \approx 3.75.$$

So

$$p_{Net}(t) \approx 1.117 \cdot t \text{ and } p_{EV}(t) \approx 1.161 \cdot t.$$

(ii) Recall that

$$p_{Var}(t) := E[S(t)] + \alpha Var[S(t)].$$

Find $\alpha > 0$ such that

$$\frac{p_{Var}(t)}{p_{EV}(t)} \underset{t \longrightarrow \infty}{\longrightarrow} 1 .$$

We have that

$$\begin{aligned}
\frac{p_{Var}(t)}{p_{EV}(t)} &= \frac{E[S(t)] + \alpha Var[S(t)]}{(1+\rho)E[S(t)]} = \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)} \frac{Var[S(t)]/t}{E[S(t)]/t} \\
&\overset{\text{Prop. 3.1.3, Mikosch}}{\longrightarrow} \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)} \frac{\lambda[Var[X_1] + Var[W_1]\lambda^2(E[X_1])^2]}{\lambda E[X_1]} \\
&= \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)} \frac{[Var[X_1] + Var[W_1]\lambda^2(E[X_1])^2]}{E[X_1]} \overset{!}{=} 1
\end{aligned}$$

for $t \longrightarrow \infty$, where $\lambda = 1/E[W_1]$. Thus

$$\alpha = \frac{\rho E[X_1]}{Var[X_1] + Var[W_1]\lambda^2(E[X_1])^2}$$

Compute the sample variance $\frac{1}{n-1}\sum_{i=1}^{n} Y_i^2 - n\overline{Y}$ of X_1 and W_1 and we get

$$Var[X_1] \approx 20.764 \text{ and } Var[W_1] \approx 5.357.$$

Then

$$\alpha \approx 0.0061.$$

Problem 4 Since the mixing variable θ is independent of $\widetilde{N}(t)$ we can treat it somewhat like a constant and get

$$E[S(t)] = E[N(t)]E[X_1] = E[\theta] \cdot t \cdot E[X_1], \tag{1}$$

where $N(t) = \widetilde{N}(\theta \cdot t)$. We see that

$$Var[S(t)] = E[(S(t))^2] - (E[S(t)])^2. \tag{2}$$

We get that

$$\begin{aligned}
E[(S(t))^2] &= E[(\sum_{i=1}^{N(t)} X_i)^2] = E[\sum_{n\geq 0}(\sum_{i=1}^{n} X_i)^2 1_{\{N(t)=n\}}] \\
&= \sum_{n\geq 0} E[(\sum_{i=1}^{n} X_i)^2 1_{\{N(t)=n\}}] \overset{N, X_i \text{ indep.}}{=} \sum_{n\geq 0} E[(\sum_{i=1}^{n} X_i)^2] \cdot P(N(t)=n) \\
&= \sum_{n\geq 0}(Var[\sum_{i=1}^{n} X_i] + (E[\sum_{i=1}^{n} X_i])^2) \cdot P(N(t)=n) \\
&\overset{X_i \text{ i.i.d}}{=} \sum_{n\geq 0}(n \cdot Var[X_1] + n^2 \cdot (E[X_1])^2) \\
&\cdot P(N(t) = n) \\
&= E[N(t)] \cdot Var[X_1] + (N(t))^2 \cdot (E[X_1])^2] \\
&= E[\theta] \cdot t \cdot Var[X_1] + (Var[N(t)] + (E[N(t)])^2)(E[X_1])^2 \\
&\overset{\text{Prob.2}}{=} E[\theta] \cdot t \cdot Var[X_1] \\
&+ (Var[\theta] \cdot t^2 + E[\theta] \cdot t + (E[\theta])^2 \cdot t^2)(E[X_1])^2.
\end{aligned}$$

So it follows from (1) and (2) that

$$\begin{aligned}
& Var[S(t)] \\
= \; & E[\theta] \cdot t \cdot Var[X_1] + \\
& (Var[\theta] \cdot t^2 + E[\theta] \cdot t + (E[\theta])^2 \cdot t^2)(E[X_1])^2 - (E[\theta])^2 \cdot t^2 (E[X_1])^2 \\
= \; & E[\theta] \cdot t \cdot (Var[X_1] + (E[X_1])^2) + Var[\theta] \cdot t^2 \cdot (E[X_1])^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{p_{Var}(t)}{p_{EV}(t)} &= \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)} \frac{Var[S(t)]}{E[S(t)]} \\
&= \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)} \frac{E[\theta] \cdot t \cdot (Var[X_1] + (E[X_1])^2) + Var[\theta] \cdot t^2 \cdot (E[X_1])^2}{E[\theta] \cdot t \cdot E[X_1]} \\
&\underset{t \longrightarrow \infty}{\longrightarrow} \infty.
\end{aligned}$$

The latter is in contrast to the renewal model for which the limit exists.