(iii) Additivity : total claims $S(t)$, $S'(t)$ indep. with premiums $P(t)$, $P'(t)$ resp.
$\longrightarrow$ $P(t)+P'(t)$ premium for $S(t)+S'(t)$

(iv) Proportionality : $c \cdot P(t)$ premium for $c \cdot S(t)$, $c>0$

If
$$P(t) = E[S(t)] \quad \text{(Equivalence principle)}$$
then (i),---,(iv) are fulfilled

$\longrightarrow$
$$P(t) = P_{Net}(t) := E[S(t)]$$
"fair market premium"
$\longrightarrow$ total loss of the insurance company becomes zero in average

Deficiency of $P_{Net}(t)$:

a) The CLT for $S(t)$ (Th. 10.1.3) shows that
$$P\left(|S(t) - E[S(t)]| \geq \sqrt{Var[S(t)]} \right) > 0$$
for large t
$\Rightarrow$ $S(t)$ deviates from $E[S(t)]$ at the order of $\sqrt{Var[S(t)]}$ for large t with positive probability

b) $P_{Net}(t)$ leads to ruin, i.e
$$P\left(u + P_{Net}(t) - S(t) < 0 \text{ for some } t>0 \right) = 1$$
for all initial capital $u>0$

$\Rightarrow$ $P_{Net}(t)$ very unwise premium strategy

3. Variance principle :
$$P_{Var}(t) := E[S(t)] + \alpha \, Var[S(t)], \quad \alpha>0$$
choose e.g. $\alpha = \delta \, \frac{E[X_1^2]}{Var[X_1] + Var[W_1]\lambda^2 (E[X_1])^2}$

Prop. 10.1.1 $\Rightarrow$
$$\frac{P_{Var}(t)}{P_{EV}(t)} \xrightarrow[t\to\infty]{} 1 \quad \Rightarrow \quad P_{Var}(t) \text{ equivalent to } P_{EV}(t)$$

4. Standard deviation principle :
$$P_{SD}(t) := E[S(t)] + \alpha \sqrt{Var[S(t)]}, \quad \alpha>0$$

Motivation :
$$P\left(S(t) - P_{SD}(t) \leq x \right) \xrightarrow[t\to\infty]{} \Phi(\alpha), \quad x \in \mathbb{R}$$
comes from the fact that
$$\frac{S(t) - E[S(t)]}{\sqrt{Var[S(t)]}} - \frac{x}{\sqrt{Var[S(t)]}} \xrightarrow[t\to\infty]{d} N(0,1) \quad \text{(Th 10.1.3)}$$

Advantage of $P_{SD}(t)$:
$$\frac{P_{SD}(t)}{P_{Net}(t)} = 1 + \alpha \, \frac{\sqrt{Var[S(t)]}}{E[S(t)]} \xrightarrow[t\to\infty]{} 1$$
($\sqrt{Var[S(t)]} \to \infty$, $E[S(t)] \to \infty$)

$\Rightarrow$ $P_{SD}(t) < P_{EV}(t), P_{Var}(t)$ for large t $\Rightarrow$ premiums can be offered at a lower price

10.3 Claim size distributions

What is a realistic model for claim size distributions?

The goodness of fit of claim size data to the model can be assessed by using e.g. QQ-plots or mean excess plots

→ 10.3.1 QQ-plots

Def. 10.3.1.1 (Quantile function)
The generalized inverse of the distr. function F, i.e.

$$F^{\leftarrow}(t) := \inf\{x \in \mathbb{R} : F(x) \geq t\}, \quad 0 < t < 1$$

is called the quantile function of F

Rem. 10.3.1.2 (i) $x_t := F^{\leftarrow}(t)$ t-quantile of F

(ii) F strictly increasing $\Rightarrow F^{\leftarrow}(t) = F^{-1}(t)$ (e.g. $F = \Phi$)

Def. 10.3.1.3 (Empirical distribution function)
Let $X_1, \dots, X_n$ be i.i.d r.v.'s. Then

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty, x]}(X_i), \quad x \in \mathbb{R}$$

is called empirical distribution function of the sample $X_1, \dots, X_n$

→ $F_n(x)$ distribution function, since

(i) $\lim_{x \to -\infty} F_n(x) = 0$ and $\lim_{x \to \infty} F_n(x) = 1$

(ii) $F_n(x) \leq F_n(y)$ for $x \leq y$

(iii) $\lim_{y \downarrow x} F_n(y) = F_n(x)$ for all $x \in \mathbb{R}$

As in Sect. 9.1.4 denote by

$$X_{(1)} \leq \dots \leq X_{(n)}$$

the order statistics of $X_1, \dots, X_n$

i.e. $X_{(j)}$ is the jth smallest of $X_1, \dots, X_n$

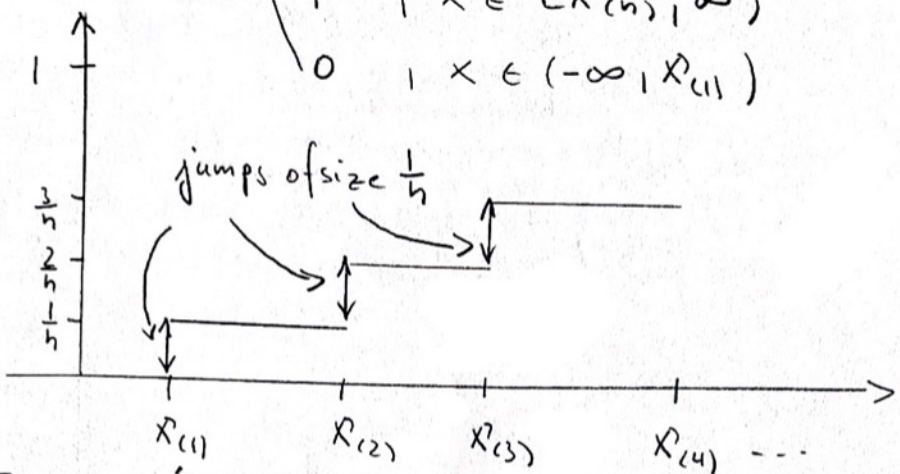
In the sequel assume that

$$X_{(1)} < \dots < X_{(n)}$$

with prob. 1 (e.g. if X_i i.i.d. with common density f)

def. of F_n $\Longrightarrow$

$$F_n(x) = \begin{cases} \frac{k}{n} & , x \in [X_{(k)}, X_{(k+1)}) \\ 1 & , x \in [X_{(n)}, \infty) \\ 0 & , x \in (-\infty, X_{(1)}) \end{cases} \quad , k = 1, \dots, n-1 \qquad (10.3.1.1)$$

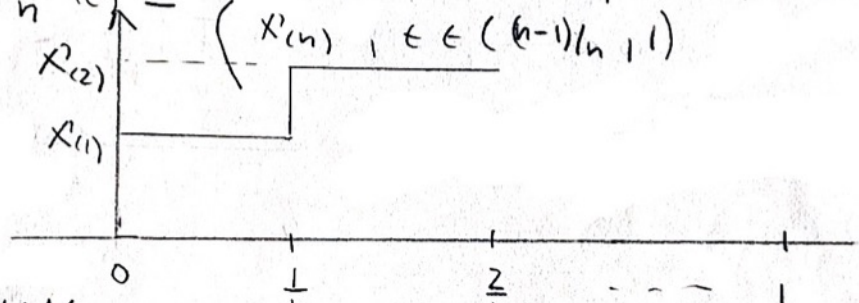


Since F_n is a distr. function we can def. the quantile function $F_n^{\leftarrow}$ of F_n

$\longrightarrow$ $F_n^{\leftarrow}$ empirical quantile function (10.3.1.2)

def. of $F^{\leftarrow}$ $\Longrightarrow$ (10.3.1.1)

$$F_n^{\leftarrow}(t) = \begin{cases} X_{(k)} & , t \in ((k-1)/n, k/n] \\ X_{(n)} & , t \in ((n-1)/n, 1) \end{cases} \quad , k = 1, \dots, n-1$$



X_i i.i.d $\overset{SLLN}{\Longrightarrow}$ $F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty, x]}(X_i) \xrightarrow[n \to \infty]{} \underbrace{E[1_{(-\infty, x]}(X_1)]}_{= P(X_1 \le x) = F(x)}$

The convergence is even uniform w.r.t. x!

Glivenko-Cantelli Lemma:

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \xrightarrow[n \to \infty]{} 0$$

$\Longrightarrow$ F_n approximation of the unknown claimsize distr. F for large n

The Glivenko-Cantelli Lemma also implies that

$$F_n^{\leftarrow}(t) \xrightarrow[n \to \infty]{} F^{\leftarrow}(t)$$

for all continuity points t of $F^{\leftarrow}$

$\Longrightarrow$ $F_n^{\leftarrow} \approx F^{\leftarrow}$ for large n

$\longrightarrow$ Conclusion: If F is a realistic model for claim size distr. then the graph of

$$(F_n^{\leftarrow}(t), F^{\leftarrow}(t)), \quad 0 < t < 1$$

will (roughly) look like a straight line for large n

$(10.3.1.2) \Rightarrow t = \frac{k}{n+1}$

The graph of

$$(x_{(k)}, F^{\leftarrow}(\tfrac{k}{n+1})), \; k = 1, \dots, n \qquad (10.3.1.3)$$

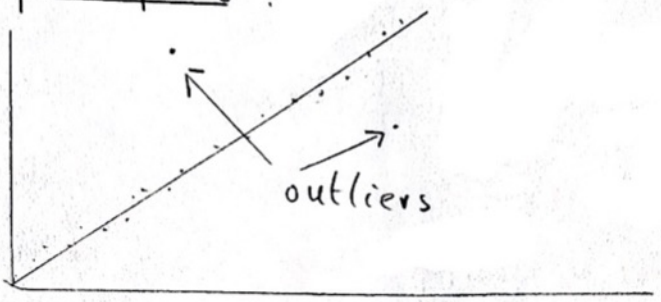
will (roughly) be a straight line

$\longrightarrow$ The graph of (10.3.1.3) is called <u>QQ-plot</u>

$\longrightarrow$ QQ-plot measure for the goodness of fit of claim size data to the model

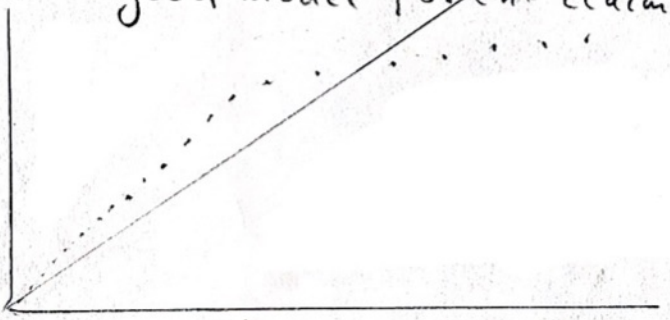
<u>Properties of QQ-plots</u>:

(i) <u>Outliers</u>:



$\longrightarrow$ F still a good model for the claim size distr.

(ii) <u>Shape</u>:



$\longrightarrow$ graph curves down at the right

$\Rightarrow$ QQ-plot indicates that the right tail of the real distr. of claim size data is heavier tailed than in the model (i.e. F)

10.3.2 Mean excess plots

Which criteria are important for a realistic modelling of claim size distributions?

$\longrightarrow$ <u>crucial criterion</u>: F <u>models</u> the probability <u>for large values of claim sizes</u>, i.e. the <u>tails</u> of the (real) claim size distr. appropriately

From the viewpoint of insurance modelling of <u>heavy-tailed claim</u> size distr. is an important issue, since higher probabilities for large claim size values in a portfolio may ruin insurance companies

One tool to discriminate heavy-tailed distr from light-tailed ones is given by

Def. 10.3.2.1 (Mean excess function)
Let $Y \geq 0$ with distr. F s.t. $E[Y] < \infty$
Set $x_\ell := \inf\{x : F(x) > 0\}$ and $x_r := \sup\{x : F(x) < 1\}$
($\Rightarrow F(x) = 0, x < x_\ell$ and $F(x) = 1, x \geq x_r$)
Then the function

$$e_F(u) := E[Y - u \mid Y > u], \quad u \in (x_\ell, x_r)$$

is called mean excess function or mean excess loss function

$\longrightarrow$ expected claim size excess over the threshold value u

$\longrightarrow$ link between e_F and the tail distribution $\bar{F} = 1 - F$:

$$e_F(u) = \frac{1}{\bar{F}(u)} \int_u^\infty \bar{F}(y)\, dy, \quad u \in [0, x_r) \qquad (10.3.2.1)$$

and

$$\bar{F}(x) = \frac{e_F(0)}{e_F(x)} \exp\left\{ - \int_0^x \frac{1}{e_F(y)}\, dy \right\}, \qquad (10.3.2.2)$$

if F continuous and $F(x) > 0, x > 0$

The def of e_F suggests to call F
(i) heavy-tailed, if $e_F(u) \xrightarrow[u \to \infty]{} \infty$
and (ii) light-tailed, if $e_F(u) \xrightarrow[u \to \infty]{} \kappa < \infty$

Ex. 10.3.2.2 ($Y \sim \text{Exp}(\lambda)$)

$$e_F(u) = \lambda^{-1}, \quad u > 0 \implies F \text{ light-tailed}$$

Ex. 10.3.2.3 (Y Pareto distr.)

$$\bar{F}(x) = 1 - F(x) = \left(\frac{\kappa}{x + \kappa}\right)^\alpha, \quad \alpha, \kappa > 0$$

$\longrightarrow$

$$e_F(u) = \frac{\kappa + u}{\alpha - 1}, \quad \alpha > 1 \implies \lim_{u \to \infty} e_F(u) = \infty$$

$\Rightarrow F$ heavy-tailed

Def. 10.3.2.4 (Empirical mean excess function)
Let $F_n(x) = \frac{1}{n} \sum_{i=1}^n 1_{(-\infty, x]}(X_i)$ the emp. distr. function
Then the mean excess function of F_n, i.e. e_{F_n} is called empirical mean excess function

$e_{F_n} \approx e_F$ for large n follows from

Prop. 10.3.2.5: Let X_i be i.i.d. claim sizes with

(56) with distr. F s.t. $E[X_1] < \infty$. Further assume that $\overline{F}(x) < 1$ for all x (i.e. $x_r = \infty$).
Then

$$e_{F_n}(u) \xrightarrow[n\to\infty]{} e_F(u)$$

with probability 1 for all $u > 0$

Proof: $e_{F_n}(u) = \frac{1}{\underbrace{P(Y>u)}_{=\overline{F}_n(u)}} E[\underbrace{1_{\{Y>u\}}}\cdot(Y-u)]$, $Y \sim F_n$
$\underbrace{\qquad\qquad}_{=(Y-u)_+ = \max(Y-u,0)}$

$\overset{\text{Ex.}}{=} \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-u)_+}{\overline{F}_n(u)} \xrightarrow{\text{SLLN}} \frac{E[(X_1-u)_+ I]}{\overline{F}(u)} = e_F(u)$

$E[(X_1-u)_+ I]$ ↙ $\overline{F}_n(u) \xrightarrow{\text{SLLN}} \overline{F}(u)$

$\Rightarrow$ proof

→ The graph of

$$(u, e_F(u)),\ u > 0$$

is "approximately" the graph of

$$(X_{(k)}, e_{F_n}(X_{(k)})),\ k = 1, \dots, n-1 \qquad (10.3.2.3)$$

→ The graph (10.3.2.3) is called mean excess plot (ME-plot)
def. of e_F

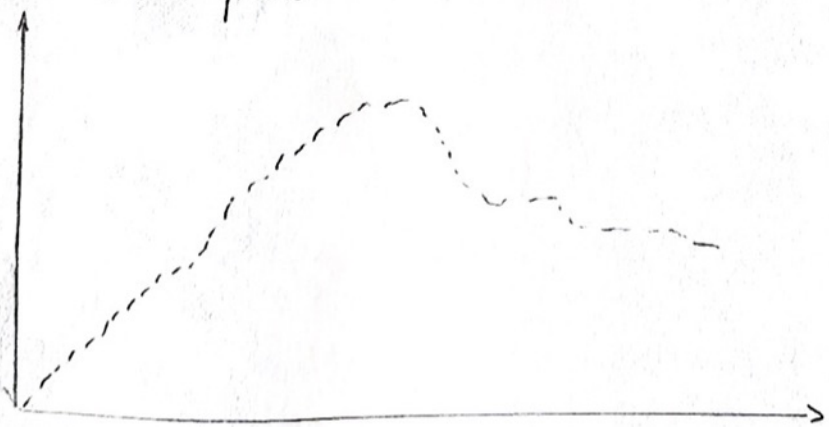
interpretation:

(i) graph of the ME-plot curves down at the right → indication that the real claim size distr. is light-tailed

(ii) graph of the ME-plot curves up at the right → indication for a heavy-tailed claim size distr.

disadvantage: This graphical method to discriminate light-tailed distr. from heavy-tailed is not very robust → better method: e.g. median excess plots

Example: ME-plot



→ graph of the ME-plot curves down at the right end

→ The ME-plot indicates that the right tail of the (real) claim size distr. is not too dangerous, i.e. light-tailed

10.4. Distribution of the total claim amount $S(t)$

Recall that

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

under our standard assumption: The claim numbers $N(t)$ are independent of the i.i.d claim sizes $X_i \geq 0$, $i \geq 1$

→ Objective: We want to study the distr. of $S(t)$

Or more generally:

Consider a decomposition

$$S(t) = S_1(t) + \ldots + S_n(t)$$

of $S(t)$ into total claim amounts $S_i(t)$, $i = 1, \ldots, n$ over certain periods of time or with respect to certain claim size layers

→ What can we say about the distributional properties of $S_i(t)$, $i = 1, \ldots, n$ of the sub-portfolios?

A useful tool for characterizing the distr. of $S(t)$ are

10.4.1 Mixture distributions

We need the following fundamental concept from probability theory:

Def. 10.4.1.1 (Characteristic function)

Let X be a r.v. Then the function

$$\phi_X(\lambda) := E[e^{i\lambda X}] := E[\cos(\lambda \cdot X)] + i\, E[\sin(\lambda \cdot X)],$$

$\lambda \in \mathbb{R}$

is called characteristic function of X

($i^2 = -1$, $|e^{i\lambda x}| = 1$)

Def. 10.4.1.2 (Multidim. characteristic function)

Let $(X_1, \ldots, X_n)$ be a random vector. Then the function

$$\phi_{X_1, \ldots, X_n}(\lambda_1, \ldots, \lambda_n) := E[e^{i(\lambda_1 X_1 + \ldots + \lambda_n X_n)}]$$
$$:= E[\cos(\lambda_1 X_1 + \ldots + \lambda_n X_n)] + i \cdot E[\sin(\lambda_1 X_1 + \ldots + \lambda_n X_n)],$$

$\lambda_1, \ldots, \lambda_n \in \mathbb{R}$

is the (n-dim.) characteristic function of $(X_1, \ldots, X_n)$

Properties of the characteristic function:

(i) Let $(X_1,\dots,X_n)$ and $(Y_1,\dots,Y_n)$ be random vectors. Then

$$(X_1,\dots,X_n) \overset{d}{=} (Y_1,\dots,Y_n)$$

$$\Longleftrightarrow \quad \phi_{X_1,\dots,X_n}(\lambda_1,\dots,\lambda_n) = \phi_{Y_1,\dots,Y_n}(\lambda_1,\dots,\lambda_n)$$

for all $\lambda_1,\dots,\lambda_n \in \mathbb{R}$

(ii) $Z_1,\dots,Z_n$ are independent

$$\Longleftrightarrow \quad \phi_{Z_1,\dots,Z_n}(\lambda_1,\dots,\lambda_n) = \phi_{Z_1}(\lambda_1)\cdot\ldots\cdot\phi_{Z_n}(\lambda_n)$$

for all $\lambda_1,\dots,\lambda_n \in \mathbb{R}$

Rem. The prob. generating function is a similar concept to $\phi_{X_1,\dots,X_n}$ for discrete r.v.'s

Notation: $S = S(t)$ and $N = N(t)$ for fixed t

Ex. 10.4.1.3 (Computation of ϕ_S)

$$\phi_S(\lambda) \overset{\text{def}}{=} E\left[e^{i\lambda\left(\sum_{i=1}^{N} X_i\right)}\right] = E\left[e^{i\lambda\left(\sum_{n\geq 0}\left(\sum_{i=1}^{n} X_i\right)\mathbb{1}_{\{N=n\}}\right)}\right]$$

$$= E\left[\sum_{n\geq 0}\left(e^{i\lambda\left(\sum_{i=1}^{n} X_i\right)}\right)\mathbb{1}_{\{N=n\}}\right] = \sum_{n\geq 0} E\left[\left(e^{i\lambda\left(\sum_{i=1}^{n} X_i\right)}\right)\cdot \mathbb{1}_{\{N=n\}}\right]$$

($e^{i\lambda\left(\sum_{i=1}^{n} X_i\right)}$ and $\mathbb{1}_{\{N=n\}}$: indep. r.v.'s)

$$= \sum_{n\geq 0} \underbrace{E\left[e^{i\lambda\left(\sum_{i=1}^{n} X_i\right)}\right]}_{=\phi_{X_1,\dots,X_n}(\lambda,\dots,\lambda)} \cdot \underbrace{E\left[\mathbb{1}_{\{N=n\}}\right]}_{=P(N=n)}$$

$$\overset{\text{Property (ii) of } \phi}{=} \sum_{n\geq 0} P(N=n)\left(\prod_{i=1}^{n}\phi_{X_i}(\lambda)\right) \overset{\text{Property (i) of } \phi}{=} \sum_{n\geq 0} P(N=n)\left(\phi_{X_1}(\lambda)\right)^n \qquad (10.4.1.1)$$

Ex. 10.4.1.4 (ϕ of a compound Poisson sum)

$N \sim \text{Pois}(\lambda) \Longleftrightarrow P(N=n) = \frac{\lambda^n}{n!}e^{-\lambda}$

$\overset{(10.4.1.1)}{\Longrightarrow}$

$$\phi_S(\lambda) = \sum_{n\geq 0} \frac{\left(\lambda\cdot\phi_{X_1}(\lambda)\right)^n}{n!} e^{-\lambda} = e^{-\lambda(1-\phi_{X_1}(\lambda))}$$

If e.g. $X_1 \sim \mathcal{U}(0,1) \Rightarrow \phi_{X_1}(\lambda) = \int_0^1 e^{i\lambda u}\,du = \frac{1}{i\lambda}e^{i\lambda u}\Big|_{u=0}^{1}$

Def. 10.4.1.5 (Mixture distribution)

Let $p_i, i=1,\dots,n$ be probabilities for $1,\dots,n$ (i.e. $\sum_{i=1}^{n} p_i = 1$) and $F_i, i=1,\dots,n$ be distr. functions of r.v.'s.

Then the distr.

$$G(x) := p_1 F_1(x) + \dots + p_n F_n(x)$$

is called *mixture distribution of $F_1,\dots,F_n$*

Rem. 10.4.1.6 Let J be a r.v. s.t. $P(J=i) = p_i$, $i=1,\dots,n$. Assume that J is indep. of r.v.'s $Y_i, i=1,\dots,n$ with $F_{Y_i} = F_i$.

Then one shows for the r.v.

$$Z := 1_{\{J=1\}} \cdot Y_1 + \ldots + 1_{\{J=n\}} Y_n$$

that

$$F_Z(x) = G(x) \text{ and } \phi_Z(x) = p_1 \phi_{Y_1}(x) + \ldots + p_n \phi_{Y_n}(x)$$

Use the method in Ex. 10.4.1.3

The following result has interesting applications in insurance

Theorem 10.4.1.7 (Sums of independent compound Poisson r.v.'s are compound Poisson)

Let

$$S_i = \sum_{j=1}^{N_i} X_j^{(i)}$$

be independent compound Poisson sums, where $N_i \sim \text{Pois}(\lambda_i)$, $\lambda_i > 0$ and $(X_j^{(i)})_{j \geq 1}$ are i.i.d claim sizes for each i.

Then

$$S = S_1 + \ldots + S_n$$

is again compound Poisson with

$$S \overset{d}{=} \sum_{i=1}^{N_\lambda} Y_i, \quad N_\lambda \sim \text{Pois}(\lambda), \quad \lambda = \lambda_1 + \ldots + \lambda_n$$

N_λ indep. of the i.i.d (Y_i) with mixture distr.

$$F_{Y_1}(x) = G(x) = \sum_{i=1}^{n} p_i \cdot F_{X_1^{(i)}}(x), \quad p_i = \lambda_i / \lambda$$

Rem.: Note that $(X_j^{(1)})_{j \geq 1}, \ldots, (X_j^{(n)})_{j \geq 1}$ are not assumed to be mutually i.i.d