Then one shows for the r.v.

$$Z := 1_{\{J=1\}} \cdot Y_1 + \ldots + 1_{\{J=n\}} Y_n$$

that $F_Z(x) = G(x)$ and $\phi_Z(x) = p_1 \phi_{Y_1}(x) + \ldots + p_n \phi_{Y_n}(x)$

Use the method in Ex. 10.4.1.3

The following result has interesting applications in insurance

Theorem 10.4.1.7 (Sums of independent compound Poisson r.v.'s are compound Poisson)

Let $S_i = \sum_{j=1}^{N_i} X_j^{(i)}$

be independent compound Poisson sums, where $N_i \sim \text{Pois}(\lambda_i), \lambda_i > 0$ and $(X_j^{(i)})_{j \geq 1}$ are i.i.d claim sizes for each i.

Then

$$\tilde{S} = S_1 + \ldots + S_n$$

is again compound Poisson with

$$\tilde{S} \overset{d}{=} \sum_{i=1}^{N_\lambda} Y_i, \quad N_\lambda \sim \text{Pois}(\lambda), \quad \lambda = \lambda_1 + \ldots + \lambda_n$$

N_λ indep. of the i.i.d (Y_i) with mixture distr.

$$F_{Y_1}(x) = G(x) = \sum_{i=1}^{n} p_i \cdot F_{X_1^{(i)}}(x), \quad p_i = \lambda_i / \lambda$$

Rem.: Note that $(X_j^{(1)})_{j \geq 1}, \ldots, (X_j^{(n)})_{j \geq 1}$ are not assumed to be mutually i.i.d

Proof:

$$\phi_{\tilde{S}}(r) \overset{\text{prop. (ii) of } \phi}{=} \phi_{S_1}(r) \cdot \ldots \cdot \phi_{S_n}(r) \overset{\text{Ex. 10.4.1.4}}{=} \prod_{i=1}^{n} e^{-\lambda_i (1 - \phi_{X_1^{(i)}}(r))} = \exp\left(-\lambda \sum_{i=1}^{n} p_i \left(1 - \phi_{X_1^{(i)}}(r)\right)\right)$$

$$= \exp\left(-\lambda \left(1 - \underbrace{\sum_{i=1}^{n} p_i \phi_{X_1^{(i)}}(r)}_{= \phi_Y(r)}\right)\right) \overset{\text{Rem. 10.4.1.6}}{=} \exp(\lambda(1 - \phi_Y(r)))$$

where $Y = \sum_{i=1}^{n} 1_{\{J=i\}} \cdot U_i$ with r.v.'s U_i s.t. $F_{U_i} = F_{X_1^{(i)}}$, $i = 1, \ldots, n$

Ex. 10.4.1.4, prop. (ii) of ϕ: $\tilde{S} \overset{d}{=} \sum_{i=1}^{N_\lambda} Y_i$ with $F_{Y_1} = F_Y$ $\Rightarrow$ proof

Ex. 10.4.1.8: Let $N(t), t \geq 0$ be claim numbers modelled by a Poisson process with mean value function $\mu(t)$. Further let $(X_j^{(i)})$ be the claim sizes in year $i \in \mathbb{N}$.

Assume that

$(X_j^{(i)})$ is i.i.d for each $i \in \mathbb{N}$

as well as

$X_j^{(i)}$ are mutually independent and indep. of N

(60) → The total claim amount in year i is given by

$$S_i := \sum_{j=N(i-1)+1}^{N(i)} X_j^{(i)}$$

One shows (Exerc. 9) that

S_i mutually indep. $= N(i)-N(i-1)$

and

$$\left(\sum_{j=N(i-1)+1}^{N(i)} X_j^{(i)} \right)_{i=1,\dots,n} \overset{d}{=} \left(\sum_{j=1}^{N(i-1,i]} X_j^{(i)} \right)_{i=1,\dots,n}$$

Set $N_i = N(i)-N(i-1)$ Then

$$S_i \overset{d}{=} \sum_{j=1}^{N_i} X_j^{(i)} ,$$

where

$$N_i \sim \text{Pois}(\mu(i-1,i])$$

$\Rightarrow$ S_i indep. compound Poisson sums

Denote by $S(n)$ the total claim amount up to n years, that is

$$S(n) = S_1 + \dots + S_n$$

Th. 10.4.1.7 $\Rightarrow$

$$S(n) \overset{d}{=} \sum_{i=1}^{N_\lambda} Y_i ,$$

where

$$N_\lambda \sim \text{Pois}(\lambda) , \lambda = \mu(0,1] + \dots + \mu(n-1,n] = \mu(n)$$

The r.v.'s Y_i are i.i.d with

$$Y_i \overset{d}{=} 1_{\{J=1\}} \cdot X_1^{(1)} + \dots + 1_{\{J=n\}} X_1^{(n)} ,$$

where the r.v. $J \in \{1,\dots,n\}$ is indep. of $X_1^{(1)}, \dots, X_1^{(n)}$

The distr. of J is given by

$$P(J=i) = \mu(i-1,i]/\lambda$$

$\Rightarrow$ $S(n)$ is compound Poisson again with i.i.d claim sizes Y_i !

Note: It was not assumed that $X_{j_1}^{(i_1)}$ has the same distr. as $X_{j_2}^{(i_2)}$ for $i_1 \neq i_2$

Ex. 10.4.1.9 : Consider n indep. portfolios with total claim amounts

$$S_i = \sum_{j=1}^{N_i} X_j^{(i)} ,$$

where $N_i \sim \text{Pois}(\lambda_i)$

and N_i indep. of $\text{c.i.d}\ (X_j^{(i)})$

for each i.

For example S_i is the total claim amount in a portfolio of car insurance policies within the tariff group (or account) i

Th. 10.4.1.7 ⟹ The aggregation of $S_i, i=1,\dots,n$, that is

$$S = S_1 + \dots + S_n$$

is compound Poisson again with counting variable

$$N \sim \text{Pois}(\lambda), \quad \lambda = \lambda_1 + \dots + \lambda_n$$

10.4.2 Space-time decomposition of the total claim amount

Now let us consider the converse problem to Th. 10.4.1.7, that is we want to study sub-portfolios of a given portfolio

More precisely, consider the total claim amount

$$S(t) = \sum_{i=1}^{N(t)} X_i = \sum_{i \geq 1} X_i 1_{[0,t]}(T_i)$$

where N is a Poisson proc. with mean value function $\mu(t)$ indep. of the c.i.d claim sizes (X_i).
T_i are the claim arrivals.
Set

$$E = [0,\infty) \times [0,\infty)$$

($[0,\infty)$: time; $[0,\infty)$: claim size values)

Let $A_1, \dots, A_n$ be a disjoint partition of E, that is

$$\bigcup_{i=1}^{n} A_i = E , \quad A_i \cap A_j = \emptyset , \quad 1 \leq i < j \leq n$$

⟶ decomposition of $S(t)$ into total claim amounts of sub-portfolios over certain periods of time and w.r.t. certain claim size layers:

$$S(t) = S_1(t) + \dots + S_n(t) ,$$

where

$$S_j(t) = \sum_{i=1}^{N(t)} X_i \cdot 1_{A_j}((T_i, X_i)) \quad , j = 1, \dots, n+1$$

Ex. 10.4.2.1 (Partitioning in time)

Choose $0 = t_0 < t_1 < \dots < t_n = t$ and set

$$\Delta_1 = [0, t_1], \quad \Delta_i = (t_{i-1}, t_i] \quad , i = 2, \dots, n, \quad \Delta_{n+1} = (t_n, \infty)$$

$\longrightarrow$ $A_i := \Delta_i \times [0, \infty)$, $i = 1, \dots, n+1$

Ex 10.4.2.2 (Partitioning in the claim size space)

Consider the different layers of claim sizes

$$B_1 = [0, d_1], \quad B_2 = (d_1, d_2], \dots, B_n = (d_{n-1}, d_n], \quad B_{n+1} = (d_n, \infty)$$

for $0 < d_1 < \dots < d_n < \infty$

$\longrightarrow$ $A_i := [0, t] \times B_i$, $i = 1, \dots, n+1$

For example different insurance companies share the risk of a portfolio by each managing sub-portfolios $S_i(t)$ w.r.t. distinct claim size magnitudes $X_j^{(i)} \in B_i$

$\longrightarrow$ converse result to Th. 10.4.1.7

Theorem 10.4.2.3 (Space-time decomposition of $S(t)$)

Let N be a Poisson proc with $\mu(t) = \int \lambda(s)\, ds$, where $\lambda > 0$ is a continuous intensity function. Further let $A_1, \dots, A_n$ be a disjoint partition of $E = [0, \infty) \times [0, \infty)$. Then

$$S_j(t) = \sum_{i=1}^{N(t)} X_i 1_{A_j}((T_i, X_i)), \quad j = 1, \dots, n$$

are independent for all $t \geq 0$. Moreover each $S_j(t)$ is compound Poisson with

$$S_j(t) \overset{d}{=} \sum_{i=1}^{N(t)} X_i 1_{A_j}((Y_i, X_i)),$$

where (Y_i) are i.i.d r.v.'s with density $\lambda(x)/\mu(t)$, $0 \leq x \leq t$, independent of N and (X_i)

Proof: More or less a direct consequence of the order statistics property of the Poisson process (Th. 9.1.4.1) see Mikosch.

10.4.3 Panjer recursion scheme

$\longrightarrow$ Exact numerical procedure for calculating the distr. of the total claim amount

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

Notation: $S = S(t)$ and $N = N(t)$ for fixed t

$$S_n := \sum_{i=1}^{n} X_i$$

$\longrightarrow$ Theorem 10.4.3.1 (Panjer recursion)

Assume that the claim sizes X_i take values in $\{0,1,2,\dots\}$ and that the claim number N has distr. given by

$$q_n := P(N=n) = \left(a + \frac{b}{n}\right) q_{n-1} \quad , n=1,2,\dots \quad (10.4.3.1)$$

for some $a, b \in \mathbb{R}$

Then the probabilities $p_n := P(S=n)$ can be calculated by

$$p_0 = \begin{cases} q_0 & , \text{if } P(X_1=0)=0 \\ E\left[\left(P(X_1=0)\right)^N\right] & \text{else} \end{cases}$$

$$p_n = \frac{1}{1-a\,P(X_1=0)} \sum_{i=1}^{n} \left(a + \frac{b\cdot i}{n}\right) P(X_1=i)\, p_{n-i} \quad , n \geq 1$$

Rem. 10.4.3.2 : Condition (10.4.3.1) is called (a,b)-condition and one shows that this cond. is only satisfied by the following 3 distributions:

1. Pois(λ) $\longrightarrow$ $a=0$, $b=\lambda \geq 0$
2. Binomial distribution $B(n,p)$
   $\longrightarrow$ $a = -p/(1-p) < 0$, $b = -a(n+1)$, $n \in \mathbb{N}_0$
3. Negative binomial distribution with param. (p,r), that is

$$P(Z=k) = \binom{r+k-1}{k} p^r (1-p)^k \quad , k \in \mathbb{N}_0, p \in (0,1), r > 0$$

$\longrightarrow$ $0 < a = 1-p < 1$ (see Ex. 9.3.2), $b = (1-p)(r-1)$, $a+b > 0$

Rem. 10.4.3.3

(i) The case $X_i \in d\mathbb{N}_0 := \{d\cdot n : n \geq 0\}$, $d > 0$ can be reduced to the case $X_i \in \mathbb{N}_0$ by rewriting $S = d \sum_{i=1}^{N} \underbrace{(X_i/d)}_{\in \mathbb{N}_0}$

(ii) Every total claim amount with continuous claim size distr. can be approximated by one with $X_i \in d\mathbb{N}_0$

Proof of Th. 10.3.4.1 : Proof based on the (a,b)-cond.

$$p_0 = P(N=0) + P(S=0, N>0)$$

If $P(X_1=0)=0 \Rightarrow p_0 = P(N=0) \overset{\text{def}}{=} q_0$

If $P(X_1=0) \neq 0 \Rightarrow$

(64)

$$p_0 = q_0 + E\Big[1_{\{\sum_{j=0}^{N} X_j^{(1)} = 0\}} \cdot 1_{\{N>0\}} \Big]$$

$$= q_0 + E\Big[\sum_{i\geq 1} 1_{\{\sum_{j=0}^{i} X_j^{(1)} = 0\}} 1_{\{N=i\}} \Big] =$$

$$q_0 + \sum_{i\geq 1} \underbrace{P(X_1^{(1)}=0, \dots, X_i^{(1)}=0)}_{= E[(P(X_1^{(1)}=0))^i]} P(N=i) \overset{\text{indep.}}{=} E\big[(P(X_1^{(1)}=0)^N\big]$$

We see that

$$p_n = \sum_{i\geq 1} P(S_i = n)\, q_i \overset{(a,b)\text{-cond.}}{=} \sum_{i\geq 1} P(S_i=n)\left(a + \frac{b}{i}\right) q_{i-1} \quad (+)$$

We have that

$$E\Big[a + \frac{b X_1^{(1)}}{n} \,\Big|\, S_i = n\Big] = a + \frac{b}{i}, \quad (++)$$

since

$$1 = E\Big[\frac{S_i}{S_i} \,\Big|\, S_i = n\Big] \overset{X_i^{(1)} \text{ i.i.d}}{=} i \cdot E\Big[\frac{X_1^{(1)}}{(S_i = n)} \,\Big|\, S_i = n\Big]$$

Further

$$E\Big[a + \frac{b X_1^{(1)}}{n} \,\Big|\, S_i = n\Big]$$

$$= \sum_{\kappa=0}^{n} \Big(a + \frac{b\cdot\kappa}{n}\Big) \underbrace{P(X_1^{(1)} = \kappa \mid S_i = n)}_{= \frac{P(X_1^{(1)}=\kappa,\, S_i - X_1^{(1)} = n-\kappa)}{P(S_i=n)}}$$

$$\overset{X_i^{(1)} \text{ i.i.d}}{=} \sum_{\kappa=0}^{n} \Big(a + \frac{b\cdot\kappa}{n}\Big) \frac{P(X_1^{(1)}=\kappa)\cdot P(S_{i-1}=n-\kappa)}{P(S_i=n)} \quad (+++)$$

$\overset{(++),(+++)}{\Longrightarrow}$

$$(+) = \sum_{i\geq 1} \sum_{\kappa=0}^{n} \Big(a + \frac{b\kappa}{n}\Big) P(X_1^{(1)}=\kappa)\, P(S_{i-1}=n-\kappa)\, q_{i-1}$$

$$= \sum_{\kappa=0}^{n} \Big(a + \frac{b\kappa}{n}\Big) P(X_1^{(1)}=\kappa) \Big(\underbrace{\sum_{i\geq 1} P(S_{i-1}=n-\kappa)\, q_{i-1}}_{= P(S=n-\kappa) \overset{\text{def}}{=} p_{n-\kappa}} \Big)$$

$$\Longrightarrow \quad p_n = a\, P(X_1^{(1)}=0)\, p_n + \sum_{\kappa=1}^{n} \Big(a + \frac{b\kappa}{n}\Big) P(X_1^{(1)}=\kappa)\, p_{n-\kappa} \quad \Longrightarrow \text{proof}$$

10.4.4 Approximation of $P(S \leq x)$ by the CLT

Consider the renewal model

$$S(t) = \sum_{i=1}^{N(t)} X_i^{(1)}, \quad t \geq 0$$

$\longrightarrow$ i.i.d claim sizes $(X_i^{(1)})$ indep. of the renewal process $N(t) = \#\{i\geq 1 : T_i \leq t\}$, where the inter-arrival times

$$W_i = T_i - T_{i-1}, \quad W_0 = 0$$

(X_i) are i.i.d.

CLT for S (Th. 10.1.3) $\Longrightarrow$

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{S(t)-E[S(t)]}{\sqrt{Var[S(t)]}} \le x \right) - \underbrace{\Phi(x)}_{\text{distr. of } N(0,1)} \right| \xrightarrow{t\to\infty} 0 \quad (10.4.4.1)$$

if $Var[W_1], Var[X_1] < \infty$

$$\Rightarrow \quad P(S(t)\le y) \approx \Phi\left((y-E[S(t)])/\sqrt{Var[S(t)]} \right)$$

for large t

E.g. $P\left(S(t) \in \left[E[S(t)] - 1.96\sqrt{Var[S(t)]},\ E[S(t)] + 1.96\sqrt{Var[S(t)]} \right] \right) \approx 0.95$

for large t

$\longrightarrow$ <u>deficiency</u>: (10.4.4.1) does not provide information about the goodness of approximation. The latter however is of high relevance for insurances when it comes to estimating e.g. the tail distr. $P(S>x)$ more accurately.

$\longrightarrow$ <u>partial solution</u>: <u>Berry-Esséen inequality</u>:

$$\sup_{x\in\mathbb{R}} \left\{ (1+|x|^3) \left| P\left(\frac{S_n - n E[X_1]}{\sqrt{n\, Var[X_1]}} \le x \right) - \Phi(x) \right| \right\} \le \frac{C}{\sqrt{n}} \frac{E[|X_1 - E[X_1]|^3]}{(Var[X_1])^3}, \quad (10.4.4.2)$$

if $E[|X_1|^3] < \infty$; Appr. of ϑ_3 ; $C \approx 30.51\ldots$ universal const

<u>Rem. 10.4.4.1</u>: Bound in (10.4.4.2) is rather sharp for discrete distr.

(10.4.4.2) $\longrightarrow$ bound of $S(t)$, conditionally on $N(t)=n(t)$:

$$\left| P\left(\frac{S(t)-n(t)E[X_1]}{\sqrt{n(t)\, Var[X_1]}} \le x \,\middle|\, N(t)=n(t) \right) - \Phi(x) \right| \le \frac{C}{\sqrt{n(t)}} \frac{1}{1+|x|^3} \frac{E[|X_1 - E[X_1]|^3]}{(Var[X_1])^3} \quad (10.4.4.3)$$

$n(t)=N(t) \xrightarrow{t\to\infty} \infty$ with prob. 1 $\Longrightarrow$ CLT gives a good approx. of $P(S(t)\le x \mid N(t)=n(t))$, if $n(t)$ is <u>large</u>

$\longrightarrow$ <u>deficiency</u>: (i) (10.4.4.3) requires sufficiently high sample sizes $n(t)$.

(ii) The bound in (10.4.4.3) is not very accurate in general

(iii) One shows for <u>fixed</u> t in the Cramér-Lundberg model that

$$P(S(t)>x) \sim \lambda t \cdot P(X_1>x) \text{ asymptotically in } x$$

for heavy-tailed X_1 (i.e. sub-exponential X_1)

$\Rightarrow$ CLT does not give a good appr. for the tail of $S(t)$

10.4.5 Approximation of $P(S \le x)$ by simulation techniques

→ e.g. Monte-Carlo simulation or bootstrap

→ computer-based methods to approximate $P(S \le x)$

1. Monte Carlo simulation:

→ idea of Monte-Carlo simulation is based on the SLLN:

Consider $S = \sum_{i=1}^{N} X_i$

with claim numbers N indep. of the i.i.d claim sizes (X_i)

Suppose $(X_i^{(1)})_{i\ge 1}, \dots, (X_i^{(m)})_{i\ge 1}$ are i.i.d copies of (X_i)

Further let $N_1, \dots, N_m$ be i.i.d copies of N

charateristic functions ⇒

$S^{(j)} := \sum_{i=1}^{N_j} X_i^{(j)}$, $j = 1, \dots, m$ i.i.d copies of S

Def. $\hat{F}_m := \frac{1}{m} \sum_{i=1}^{m} \underbrace{1_{(-\infty, x]}(S^{(i)})}_{=: Z_i \text{ i.i.d}}$

SLLN → $\hat{F}_m \xrightarrow{m\to\infty} E[Z_1] = E[1_{(-\infty,x]}(S^{(1)})] = P(S \le x) = F_S(x)$

$\Rightarrow \hat{F}_m \approx F_S(x)$ for large m

→ In practice:

1. step: Draw raud. numbers from the raud. numb. generator for the distr. of N m times

→ Output: $N_1, \dots, N_m$

2. step: Draw raud. numb. from the raud. numb. gen. for the distr. of X_1 N_j times for each $j = 1, \dots, m$

→ Output:

$$\begin{matrix} X_1^{(1)}, \dots, X_{N_1}^{(1)} \\ \vdots \\ X_1^{(m)}, \dots, X_{N_m}^{(m)} \end{matrix}$$

→ $S^{(1)}, \dots, S^{(m)}$

3. step: Plug the values $S^{(1)}, \dots, S^{(m)}$ into $\hat{F}_m$

→ Output: $\hat{F}_m \approx F_S(x)$