(9) $\xrightarrow{t=0,\ T=0}$ $F(0) = e^{-\int_0^h \delta(s)\,ds} F(h)$

present value of a payment $F(h)$ to be made at time h

→ Application: Calculation of the present value of various types of payment streams

2.3 Perpetuities

Perpetuities: perpetual payment streams, i.e. cash flows without expiry ($T=\infty$)

Present value of the following perpetuities with annual payments of 1 unit.

Ex 2.3.1 (perpetuity-due)

1. payment at $t=0$

→ present value (PV): $\ddot{a}_{\overline{\infty}|} := \sum_{k\geq 0} v^k = \frac{1}{1-v}$

Ex. 2.3.2 (immediate perpetuity)

1. payment at $t=1$ year

→ PV: $a_{\overline{\infty}|} := \sum_{k\geq 1} v^k = v\left(\sum_{\ell\geq 0} v^\ell\right) = \frac{v}{1-v} = \frac{1}{i}$

Ex. 2.3.3 (perpetuity-due with payment $\frac{1}{m}$ m times per year

→ PV: $\ddot{a}^{(m)}_{\overline{\infty}|} := \sum_{k\geq 0} \frac{1}{m} v^{\frac{k}{m}} = \frac{1}{m}\left(\sum_{k\geq 0} \left(v^{\frac{1}{m}}\right)^k\right)$

$= \frac{1}{m}\,\frac{1}{1-v^{\frac{1}{m}}}$

Ex. 2.3.4 (continuous perpetuity)

"continuous" means: $m \to \infty$ in Ex. 2.3.3

→ PV: $\bar{a}_{\overline{\infty}|} := \lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{\infty}|}$

Riemann sum $\int_0^\infty v^t\,dt = \int_0^\infty e^{-\delta t}\,dt = \frac{1}{\delta}$

Ex 2.3.5: perpetuity with payment stream $r_0, r_1, r_2, \ldots$ at time $t = 0, 1, 2, \ldots$

→ PV: $\ddot{a} := \sum_{k\geq 0} r_k v^k$

2.4 Annuities

Annuities: payment streams of (limited duration ($T < \infty$))

Ex. 2.4.1 (annuity-due)

n annual payments of 1 starting in $t = 0$

$\longrightarrow$ PV: $\ddot{a}_{\overline{n}|} := 1 + v + v^2 + \ldots + v^{n-1}$

Note that $\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{\infty}|} - v^n \ddot{a}_{\overline{\infty}|}$

perp. starting in zero; perp. start. in n

Further one shows that

$a_{\overline{n}|} = \frac{1 - v^n}{i}$ (immediate annuity)

$a^{(m)}_{\overline{n}|} = \frac{1 - v^n}{i^{(m)}}$ (immediate annuity, $\frac{1}{m}$ m times)

accumulated value of annuities:

$\ddot{s}_{\overline{n}|} := (1+i)^n \cdot \ddot{a}_{\overline{n}|}$ (accumulated annuity-due)

$s_{\overline{n}|}$, $\ddot{s}^{(m)}_{\overline{n}|}$, $s^{(m)}_{\overline{n}|}$

$\longrightarrow \frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$ (see Exerc. 1)

(11)

Now we turn our attention to stochastic cash flows, i.e. to payment streams with payments subject to a random source of risk
Let us first examine the rand. source of risk given by mortality

→ random source of risk: mortality

→ objective: model for this rand. source of risk

3. The future lifetime of a life aged x

3.1 Model

Notation: (x) life aged x years

$T = T(x)$ future lifetime of the person

→ $x+T$ age at death

model for T:

T random variable on a probability space $(\Omega, \mathcal{F}, P)$

1. Ω: sample space: set of all possible outcomes of a random experiment

Ex.: Roll a die → $\Omega = \{1, 2, 3, \dots 6\}$

2. $\mathcal{F}$: σ-algebra (in Ω): collection of events $A \in \mathcal{F}$ (e.g. of market events)

$\mathcal{F}$ is a family of subsets of Ω s.t.

a) $\emptyset \in \mathcal{F}$

b) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ $\quad (A^c := \Omega \setminus A)$

c) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i \geq 1} A_i \in \mathcal{F}$

Ex.: $\Omega = \{1, 2, \dots, 6\}$, $\mathcal{F} := \mathcal{P}(\Omega)$ class of all subsets of Ω

$A := \{1, 2, 3\} \in \mathcal{F}$ event that 1, 2 or 3 occurs

3. P: Probability measure: Function $P: \mathcal{F} \to [0, 1]$ s.t. (impossible event, sure event)

a) $P(\emptyset) = 0$, $P(\Omega) = 1$

b) $A_1, \dots, A_n \in \mathcal{F}$ with $A_i \cap A_j = \emptyset$, $i \neq j$ (disjoint)

$\Rightarrow P\left(\bigcup_{i \geq 1} A_i\right) = \sum_{i \geq 1} P(A_i)$

$P(A)$ = "the prob. that the event A occurs"

with e.g. A = "event that the stock price exceeds 100 $"

Ex.: $\Omega = \{1,\dots,6\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, $P(A) := \frac{1}{6}\#A$

T random variable, that is

$T : \Omega \to \mathbb{R}$ function s.t.

$\{T \le t\} := \{\omega \in \Omega : T(\omega) \le t\} \in \mathcal{F}$ (collect. of all events)

for all t, i.e.

"(x) dies within t years" is a possible event for all t

→ reasonable assumption!

assumptions and notation:

$\Pr := P$

$G(t) := \Pr(\{T \le t\})$, $t \ge 0$

→ prob. that (x) dies within t years

G continuous, i.e.

$G'(t) = g(t)$ probability density

$\iff$: $\Pr(t < T < t+dt) = g(t)\,dt$

→ actuarial notation:

$${}_tq_x := G(t) \qquad (3.1.1)$$

$${}_tp_x := 1 - G(t) = \Pr(T > t) \left(= \Pr(\{\omega \in \Omega : T(\omega) > t\})\right)$$

→ Prob. that (x) survives at least t years

$${}_{s|t}q_x := \Pr(s < T < s+t) = G(s+t) - G(s) \qquad (3.1.2)$$

$$= {}_{s+t}q_x - {}_sq_x$$

→ Prob. that (x) survives s years and dies within t years (after)

$${}_tp_{x+s} := \Pr(T > s+t \mid T > s) = \frac{1 - G(s+t)}{1 - G(s)} \qquad (3.1.3)$$

→ prob. that (x) survives another t years, after having reached $x+s$ years

$${}_tq_{x+s} := \Pr(T \le s+t \mid T > s) = \frac{G(s+t) - G(s)}{1 - G(s)} \qquad (3.1.4)$$

Expected remaining lifetime of (x):

$$\mathring{e}_x := E[T] = \int_0^\infty t\, g(t)\,dt$$

or

$$= \int_0^\infty (1 - G(t))\,dt \qquad (3.1.5)$$

$$= \int_0^\infty {}_tp_x\,dt$$

shorthand notation: $q_x := {}_1q_x$, $p_x := {}_1p_x$, ${}_{s|}q_x := {}_{s|1}q_x$

3.2 Force of mortality (dødsintensitet)

force of mortality of (x) at $x+t$:

$$\mu_{x+t} := \frac{g(t)}{1-G(t)} = -\frac{d}{dt}\log(1-G(t))$$

→ measure for the speed of mortality

Since $g(t) = \underbrace{{}_tp_x}_{1-G(t)} \cdot \mu_{x+t}$

(3.1.5) implies that

$$\mathring{e}_x = \int_0^\infty t \; {}_tp_x \, \mu_{x+t} \, dt \qquad (3.2.1)$$

Further, since

$$\mu_{x+t} = -\frac{d}{dt}\log {}_tp_x$$

we get by integration that

$${}_tp_x = e^{-\int_0^t \mu_{x+s}\, ds} \qquad (3.2.2)$$

Rem. 3.2.1 : Because of (3.1.4) we observe that

$${}_sq_{x+t} \approx \mu_{x+t}\text{'s} \quad \text{for small } s$$

Now let us discuss some explicit distributions for T

3.3 Analytical distributions of T

→ explicit formula for G_T

→ favourable from a computational point of view

Examples :

1. **de Moivre (1724)** :

$$g(t) = \frac{1}{\omega - x}, \quad 0 < t < \omega - x \quad \text{(uniform distr. on } [0, \omega-x]\text{)}$$

ω maximum age

$$\Rightarrow \quad \mu_{x+t} \overset{\text{def}}{=} \frac{g(t)}{1-\underbrace{G(t)}_{\frac{t}{\omega-x}}} = \frac{1}{\omega - x - t}, \quad 0 < t < \omega - x$$

2. **Gompertz (1824)** :

$$\mu_{x+t} := B\, c^{x+t}, \quad t > 0$$

3. **Makeham (1860)** :

$$\mu_{x+t} := \underbrace{A}_{\geq 0} + B\, c^{x+t}, \quad t > 0$$

indep. of age

$$\overset{(3.2.2)}{\Longrightarrow} \quad {}_tp_x = \exp\left(-At - m\, c^x (c^t - 1)\right), \quad m = B/\log c$$

4. **Weibull (1939)** :

$$\mu_{x+t} = \underbrace{K}_{>0} (x+t)^n > 0$$

For later use we have to introduce some r.v.'s related to T

3.4 Curtate future lifetime of (x)

1. K curtate future lifetime of (x):

$$K := [T]$$ ← Gauss bracket; rounds off to the next lower integer

e.g. $[1.7] = 1$, $[2.9] = 2$, $[3.1] = 3$

→ # of completed future years lived by (x)

def ⇒ $\Pr(K = j) = \Pr(j \le T < j+1) = {}_jp_x \cdot q_{x+j}$, (3.4.1)

${}_jp_x$ ← surv. prob.; q_{x+j} ← cond. prob. to die within 1 year, after having reached age $x+j$ years

$j \in \mathbb{N}_0$

expected value of K:

$$e_x := E[K] = \sum_{j \ge 1} j \Pr(K=j) \overset{(3.4.1)}{=} \sum_{j \ge 1} j \cdot {}_jp_x \cdot q_{x+j} \quad (3.4.2)$$

or

$$e_x = \sum_{j \ge 1} \Pr(K \ge j) = \sum_{j \ge 1} {}_jp_x$$

2. S:

$$S := T - K \in [0,1)$$

Rem. 3.4.1: (i) S has cont. distr. on $[0,1]$ ← remaining lifetime in the year of death

(ii) $\mathring{e}_x \approx e_x + \frac{1}{2}$

3. $S^{(m)}$:

$$S^{(m)} := \frac{1}{m} [m \cdot S + 1]$$ ← Gauss bracket

rounds to the next higher multiple of $\frac{1}{m}$

e.g. $S = \frac{2}{m} + \frac{1}{2m} \rightarrow S^{(m)} = \frac{3}{m}$

Next we want to give an approximation of the distribution of T by means of one-year survival probabilites

3.5 Approximation of $G(t)$

Consider 1-year survival prob.

p_{x+j}, $j = 0,1,2,\dots$

given by a life table (see later)

→ ${}_kp_x$, $k \in \mathbb{N}$ known, since

$${}_kp_x = p_x \, p_{x+1} \, p_{x+2} \cdots p_{x+k-1}, \quad k \in \mathbb{N}$$

→ objective: approximation of $G(t) = \Pr(T \le t)$

→ method: interpolation of ${}_uq_x$ or μ_{x+u} at intermediate ages $x+u$, $x \in \mathbb{N}_0$, $0 < u < 1$

e.g. if μ_{x+u} known by interpolation

$\Rightarrow {}_tp_x \overset{(3.2.2)}{=} e^{-\int_0^t \mu_{x+s}\,ds}$ known for all t

$\Rightarrow G(t) = 1 - {}_tp_x$ known for all t

1. method:

assumption: $({}_uq_{x+j} \overset{!!!}{=} u\,q_{x+j})$

$${}_uq_x = u \cdot q_x \,, \quad 0 \le u \le 1 \qquad \text{(linearity of } {}_uq_x\text{)}$$

$$\Rightarrow {}_up_x = 1 - u q_x$$

$$\overset{\text{def.}}{\Rightarrow} \quad \mu_{x+u} = \frac{q_x}{1 - u \cdot q_x} \qquad (3.5.1)$$

Rem. 3.5.1: (3.5.1) holds, if e.g. S uniformly distributed and K, S are indep. (see Exc 2)

2. method:

assumption: $\mu_{x+u} = \mu_{x+\frac{1}{2}} \,, \quad 0 < u < 1$

We now consider stochastic payment streams depending on the future lifetime T

4. Life insurance

An insurance policy is specified by

1. benefits payable by the insurer
2. premiums payable by the insured

Assumption: benefits and premiums may depend on the r.v. T

$Z :=$ PV of the benefits

$V :=$ PV of the premium payments

$\Rightarrow$ Z, V r.v.'s

$\longrightarrow$ $L := Z - V$ total loss of the insurer

$\longrightarrow$ reasonable to require that

$$E[L] = 0$$

$\Leftrightarrow$ $$E[V] = E[Z]$$

$\longrightarrow$ $E[Z]$ net single premium

Remark 4.1: $E[Z]$ is not a measure for the risk of the insurer

$\longrightarrow$ further characteristics of the distribution of Z (e.g. tails of distributions) are required for the assessment of risk

Examples of insurance policies based on T:

4.1 Elementary insurance types

4.1.1 Whole life and term insurance

1. whole life insurance (Engl. Lebensversicherung)

$$Z := v^{K+1} \quad \text{at } t = K+1$$

$$K = [T]$$

$$v = \frac{1}{1+i}.$$

$\longrightarrow$ payment 1 at the end of the year of death

(17) $\longrightarrow$ net single premium:

$$A_x := E[Z] = E[v^{K+1}]$$

$$= \sum_{j \geq 0} v^{j+1} \Pr(K = j)$$

$$\overset{(3.4.1)}{=} \sum_{j \geq 0} v^{j+1} \, {}_jp_x \, q_{x+j}$$

$\longrightarrow$ "risk measure" $\mathrm{Var}[Z]$:

$$\mathrm{Var}[Z] = E[Z^2] - A_x^2$$

$v = e^{-\delta}$, δ force of interest

$\Rightarrow$ $E[Z^2] = E[e^{-2\delta(K+1)}] =$

$$\sum_{j \geq 0} e^{-2\delta(j+1)} \, {}_jp_x \, q_{x+j}$$