→ net single premium:

$$A_x := E[Z] = E[v^{K+1}]$$

$$= \sum_{j \geq 0} v^{j+1} \Pr(K = j)$$

$$\overset{(3.4.1)}{=} \sum_{j \geq 0} v^{j+1} \, {}_jp_x \, q_{x+j}$$

→ "risk measure" $Var[Z]$:

$$Var[Z] = E[Z^2] - A_x^2$$

$v = e^{-\delta}$, δ force of interest

$$\Rightarrow E[Z^2] = E[e^{-2\delta(K+1)}] =$$

$$\sum_{j \geq 0} e^{-2\delta(j+1)} \, {}_jp_x \, q_{x+j}$$

2. term insurance (risiko- eller. korttidsforsikring)

$$Z = v^{K+1} \cdot 1_{\{K<n\}} \quad \text{at } t = K+1$$

→ payment of 1, if year of death $< n$ duration of the contract otherwise zero

→ net sing. prem:

$$A^1_{x:\overline{n}|} := E[Z] = \sum_{j=0}^{n-1} v^{j+1} \, {}_jp_x \, q_{x+j}$$

4.1.2 Pure endowments (opplevelsesforsikringer)

pure endowment of duration n:

$$Z = v^n \cdot 1_{\{K \geq n\}} \quad \text{at } t = n$$

→ paym. of 1, if the insured is alive at the end of n years else zero

→ net sing. prem.:

$$A_{x:\overline{n}|}^{1} := E[Z] = v^n \, {}_np_x$$

4.1.3 Endowments (sammensatt livsforsikring)

→ mixed case

→ Ex. 4.1.3.1:

$$Z = \underbrace{v^{K+1} 1_{\{K<n\}}}_{=:Z_1} + \underbrace{v^n \cdot 1_{\{K \geq n\}}}_{=:Z_2} \quad \text{at } t = K+1, n$$

Z_1: term insur. Z_2: pure endowment

→ net single prem.:

$$A_{x:\overline{n}|} := E[Z] = E[Z_1] + E[Z_2]$$

$$= A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{1}$$

4.2 Payment at the moment of death

Consider e.g.

$Z = v^T$ at $t = T$

$\longrightarrow$ paym. of 1 at $t=T$, i.e. at the instant of death

$\longrightarrow$ net sing. prem.: $\overline{A}_x := E[Z] = \int_0^\infty v^t \underbrace{{}_tp_x \mu_{x+t}}_{g(t)} dt$

4.3 General types of life insurance

We consider benefits depending on time:

1. c_j payments at j-th year:

$Z = c_{K+1} v^{K+1}$

$\Rightarrow E[Z] = \sum_{j \geq 0} c_{j+1} v^{j+1} {}_jp_x \cdot q_{x+j}$

2. $c(t)$ cont. paym.:

$Z = c(T) v^T$

$\Rightarrow E[Z] = \sum_{j \geq 0} E[Z \mid K=j] \cdot Pr(K=j)$

$T = K+S$

$= \sum_{j \geq 0} E[c(j+S) v^{j+S} \mid K=j] \cdot \underbrace{Pr(K=j)}_{= {}_jp_x \cdot q_{x+j}}$ (see Exerc. 2, Prob. 8)

4.4 Equations for A_x, $\overline{A}_x$

net sing. pr. of a whole life ins.

1. A_x: $A_x = E[v^{K+1}] = \sum_{j \geq 0} v^{j+1} {}_jp_x q_{x+j}$

${}_jp_x = p_x \cdot {}_{j-1}p_{x+1}$

$\Rightarrow A_x = v^{0+1} \underbrace{{}_0p_x}_{=1} \cdot q_{x+0} + \sum_{j \geq 1} \underbrace{v^{j+1} {}_jp_x q_{x+j}}_{= v^{j+1} p_x \cdot {}_{j-1}p_x \cdot q_{x+j}}$

$= v^{j+1} p_x \cdot {}_{j-1}p_x \cdot q_{x+j}$

$= v \cdot v^{j-1}$

$= q_{(x+1)+(j-1)}$

$= v \cdot q_x + v \cdot p_x \underbrace{\sum_{j \geq 0} v^j {}_jp_{x+1} \cdot q_{x+1+j}}_{= A_{x+1}}$

$\Rightarrow A_x = v q_x + v A_{x+1} p_x$ (4.4.1)

$= E[Y]$,

where $Y = \begin{cases} v^{a\pi}, & \text{if } T \leq 1 \\ v A_{x+1}, & T > 1 \end{cases}$

(4.4.1) $\longrightarrow$

$A_x = v A_{x+1} + v(1 - A_{x+1}) q_x = \underbrace{v A_{x+1}}_{\text{reserve}} + \underbrace{A^1_{x\pi} \cdot (1 - A_{x+1})}_{\text{net sing. prem. of a term ins. with 1-year term and payment } (1 - A_{x+1})}$

2. $\bar{A}_x$: $\bar{A}_x = E[v^T] = $ (net sing. prem. of the same person aged x+h) $= \bar{A}_{x+h}$

$$E[v^T \mid T \le h] \cdot {}_hq_x + v^h E[v^{T-h} \mid T > h] \cdot {}_hp_x$$

$$\Rightarrow \frac{\bar{A}_{x+h} - \bar{A}_x}{h} = \frac{(1 - v^h \cdot {}_hp_x)}{h} \bar{A}_{x+h} - \frac{E[v^T \mid T \le h] \cdot {}_hq_x}{h}$$

$h \to 0$: $\frac{(1 - v^h \cdot {}_hp_x)}{h} \to \delta + \mu_x$; $\bar{A}_{x+h} \to \bar{A}_x$; $\frac{E[v^T \mid T \le h] \cdot {}_hq_x}{h} \to g(0) = \mu_x$

$$\Rightarrow \delta \cdot \bar{A}_x = \frac{d}{dx} \bar{A}_x + \mu_x (1 - \bar{A}_x) \qquad (4.4.2)$$

5. Life annuities

Life annuities ⟶ e.g. annuity-due, immediate annuity depending on the future life time T

Examples ⟶

5.1 Elementary life annuities

Ex 5.1.1 : whole life annuity-due (straks begynnende annuitet av varighet K+1):

$$Y = \sum_{j=0}^{K} v^j = \ddot{a}_{\overline{K+1}|}$$

⟶ PV of annual paym. of 1 until n=K (starting in zero)

⟶ net sing. prem.:

$$\ddot{a}_x := E[Y] = \sum_{j \ge 0} \ddot{a}_{\overline{j+1}|} \underbrace{Pr(K=j)}_{= {}_jp_x \cdot q_{x+j}}$$

Rewrite Y as

$$Y = \sum_{j \ge 0} v^j \cdot \underbrace{1_{\{K \ge j\}}}_{\text{PV of a pure endowment}}$$

E[·] ⟹ $\ddot{a}_x = \sum_{j \ge 0} A^{\frac{1}{}}_{x\,\overline{j}|}$

Ex. 5.1.2 : n-year temporary life annuity-due

$$Y = \ddot{a}_{\overline{K+1}|} \cdot 1_{\{K<n\}} + \ddot{a}_{\overline{n}|} \cdot 1_{\{K \ge n\}}$$

⟶ PV of an whole life annuity-due, if K<n duration or of an annuity-due, if K≥n

⟶ net sing. prem.

$$\ddot{a}_{x:\overline{n}|} := E[Y] = \sum_{j=0}^{n-1} \ddot{a}_{\overline{j+1}|} \, {}_jp_x \, q_{x+j} + \ddot{a}_{\overline{n}|} \cdot {}_np_x$$

Ex. 5.1.3 : immediate life annuity:

$$Y = \sum_{j=1}^{K} v^j = a_{\overline{K}|}$$

⟶ PV of paym. of 1 from t=1 to t=K

5.2 Life annuity with m payments per year

$$Y = \ddot{a}^{(m)}_{\overline{K+S^{(m)}|}}$$

$\longrightarrow$ PV of paym. $\frac{1}{m}$ m times per year from $t=0$ to $t = K + S^{(m)} - \frac{1}{m}$; $S^{(m)} = \frac{1}{m}[mS+1]$; m-th fraction of the year in which the insured dies

$\longrightarrow$ net sing. prem.

$$\ddot{a}^{(m)}_x = E[Y]$$

Rem. 5.2.1 : If K, S indep., S unif. distr. on $[0,1]$ then

$$\ddot{a}^{(m)}_x = \alpha(m)\,\ddot{a}_x - \beta(m)$$

where $\alpha(m) \approx 1$ and $\beta(m) \approx \frac{m-1}{2m}$ (see Exerc. 2)

5.3 Variable life annuities

$$Y = \sum_{j=0}^{K} r_j v^j = \sum_{j\geq 0} r_j v^j 1_{\{K \geq j\}}$$

$\longrightarrow$ PV of paym. of $r_1, \dots, r_K$

$\longrightarrow$ net sing. prem.

$$E[Y] = \sum_{j\geq 0} r_j v^j {}_jp_x$$

Consider the continuous-time case:

$r(t)$, $t \geq 0$ paym. rate at time t

$\longrightarrow$ $Y = \int_0^T v^t r(t)\,dt$

$\longrightarrow$ net sing. prem.

$$E[Y] = \int_0^\infty v^t r(t)\, {}_tp_x\, dt$$ (see Exerc. 3)

5.4 Equations for $\ddot{a}_x$, $\overline{a}_x$

1. $\ddot{a}_x$: From

$$\ddot{a}_x = \sum_{j\geq 0} v^j {}_jp_x$$

and ${}_jp_x = p_x \cdot {}_{j-1}p_{x+1}$

it follows

$$\ddot{a}_x = 1 + v\,\ddot{a}_{x+1}\cdot p_x \qquad (5.4.1)$$

$\Longleftrightarrow$ $\ddot{a}_x = \underbrace{1 + v\,\ddot{a}_{x+1}}_{\text{bond}} - \underbrace{v\,\ddot{a}_{x+1} q_x}_{\text{expected mortality gain}}$

2. $\bar{a}_x$:

$$\bar{a}_x := \lim_{m\to\infty} \ddot{a}_x^{(m)} \qquad \ddot{a}_x^{(m)} := E[v^{K+S^{(m)}}] \xrightarrow[m\to\infty]{} \bar{A}_x = E[v^T]$$

(S for $m \to \infty$)

One shows that

$$1 = \underbrace{m\left(1-(1+i)^{-1/m}\right)}_{\xrightarrow[m\to\infty]{}\delta} \ddot{a}_x^{(m)} + A_x^{(m)} \qquad (5.4.2)$$

$$\Rightarrow \quad 1 = \delta \bar{a}_x + \bar{A}_x \qquad (5.4.3)$$

(4.4.2)

$$\Rightarrow \quad \delta \bar{a}_x = 1 + \frac{d}{dx}\bar{a}_x - \mu_x \bar{a}_x$$

5.5 Calculation of $\ddot{a}_{x+u}$, $x \in \mathbb{N}_0$, $0 < u < 1$

Observe that

$${}_uP_x \cdot {}_kP_{x+u} = {}_kP_x \, {}_uP_{x+k}$$

Ass. (see Sect 3.5) : ${}_uq_{x+j} = u \cdot q_{x+j}$, $0 < u < 1$

$$\Rightarrow (1-u\cdot q_x)\, {}_kP_{x+u} = {}_kP_x \cdot (1-u\cdot q_{x+k})$$

Multiplication with v^k and summation over k give

$$(1-u\cdot q_x)\cdot \ddot{a}_{x+u} = \ddot{a}_x - u(1+i)A_x \quad \leftarrow \text{recursion for } \ddot{a}_x$$

Then, using (5.4.2) for $m=1$ and (5.4.1) we get

$$\ddot{a}_{x+u} = \frac{1-u}{1-u\cdot q_x}\ddot{a}_x + \frac{u(1-q_x)}{1-u\cdot q_x}\ddot{a}_{x+1}$$

$$\Rightarrow \quad \ddot{a}_{x+u} \approx (1-u)\ddot{a}_x + u\cdot \ddot{a}_{x+1},$$

if q_x small