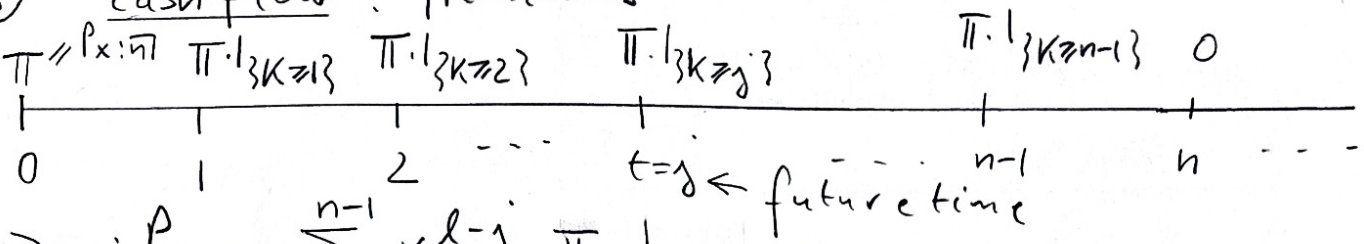
(26) cash flow: premiums



$$\longrightarrow {}_jP = \sum_{\ell=j}^{n-1} v^{\ell-j} \cdot \Pi \cdot 1_{\{K\geq \ell\}}$$

$$\Rightarrow E[{}_jP \mid T>j] = \sum_{\ell=j}^{n-1} v^{\ell-j} \cdot \Pi \cdot \underbrace{E[1_{\{K\geq \ell\}} \mid T>j]}_{= Pr(K\geq \ell \mid T>j) = {}_{\ell-j}p_{x+j}}$$

$$= \sum_{\ell=j}^{n-1} v^{\ell-j} \Pi \, {}_{\ell-j}p_{x+j} = \sum_{\mu=0}^{n-1-j} v^{\mu} \cdot \Pi \cdot {}_\mu p_{x+j}$$

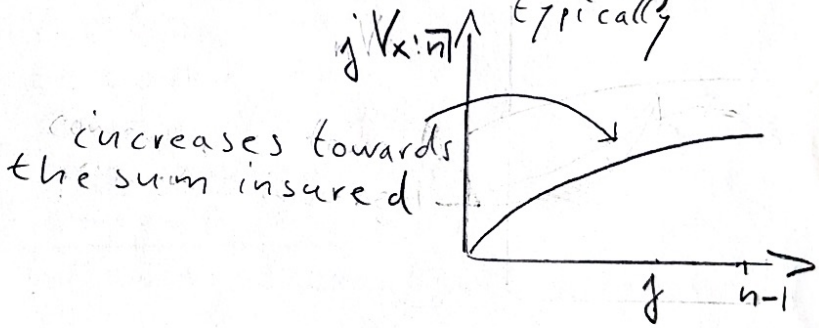
$$= \underbrace{P}_{=\Pi}{}_{x:\overline{n}|} \cdot \ddot{a}_{x+j:\overline{n-j}|} \quad \leftarrow \text{net sing. pr. of a temp. life annuity}$$

$$\Rightarrow {}_jV_{x:\overline{n}|} := {}_jV = E[\underbrace{{}_jL}_{\leq {}_jZ - {}_jP} \mid T>j] = E[{}_jZ \mid T>j] - E[{}_jP \mid T>j]$$

$$= A_{x+j:\overline{n-j}|} - P_{x:\overline{n}|} \cdot \ddot{a}_{x+j:\overline{n-j}|} \qquad (7.1.3)$$

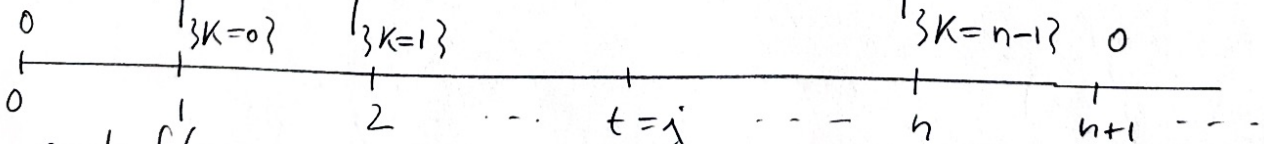
$\xrightarrow[\text{in (7.1.3)}]{j=n-1}$ ${}_{n-1}V_{x:\overline{n}|} = v - P_{x:\overline{n}|} \iff (1+i)({}_{n-1}V_{x:\overline{n}|} + P_{x:\overline{n}|}) = 1$

$\longrightarrow$ interest earned on ${}_{n-1}V$ plus the premium is used to cover the sum insured (i.e. 1)



Ex. 7.1.3 (term insurance)

cash flow: benefits



cash flow: premiums as in Ex 7.1.2

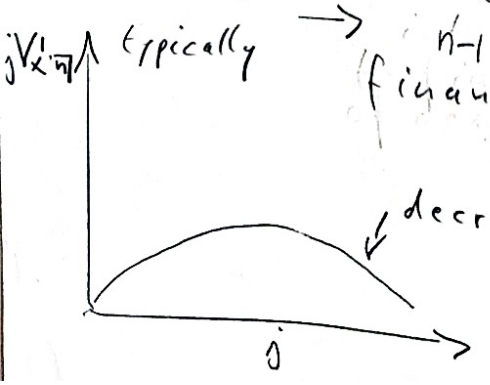
same reasoning as in Ex 7.1.2 $\xrightarrow[\text{in (7.1.4)}]{j=n-1}$

$$(7.1.4) \qquad {}_jV^1_{x:\overline{n}|} := {}_jV = A^1_{x+j:\overline{n-j}|} - P^1_{x:\overline{n}|} \, \ddot{a}_{x+j:\overline{n-j}|}$$

$\leftarrow$ NSP of a term insurance

$$({}_{n-1}V^1_{x:\overline{n}|} + P^1_{x:\overline{n}|}) = A^1_{x+n-1:\Pi} \quad (= v \cdot q_{x+n-1})$$

$\longrightarrow$ ${}_{n-1}V$ plus the premium is used to finance the NSP of a 1-year term insurance



7.2 Recursion formulas for ${}_jV$

Consider the gen. life insurance of Section 4.3 with benefits C_j and premiums Π_j

$$\longrightarrow \quad L = C_{K+1} v^{K+1} - \sum_{j=0}^{K} \Pi_j \cdot v^j$$

Rem. 7.2.1 : This model is quite general and includes various endowments and life annuities
e.g. an endowment is obtained, cf
and $C_1 = C_2 = \ldots = C_n = 1$, $C_{n+1} = C_{n+2} = \ldots = 0$
$\Pi_0 = \Pi_1 = \ldots = \Pi_{n-1} = P_{x:\overline{n}|}$, $\Pi_n = -1$, $\Pi_{n+1} = \Pi_{n+2} = \ldots = 0$

As in Ex. 7.1.2 one shows that

$$ {}_KV = \sum_{j \geq 0} C_{K+j+1} v^{j+1} {}_jp_{x+K} q_{x+K+j} - \sum_{j \geq 0} \Pi_{K+j} v^j {}_jp_{x+K} \tag{7.2.1}$$

Using $ {}_jp_{x+K} = {}_hp_{x+K} \cdot {}_{j-h}p_{x+K+h}$, $j \geq h$

in (7.2.1) one finds that

$$ {}_KV + \sum_{j=0}^{h-1} \Pi_{K+j} v^j {}_jp_{x+K} = \sum_{j=0}^{h-1} C_{j+K+1} v^{j+1} {}_jp_{x+K} q_{x+K+j} + {}_hp_{x+K} v^h {}_{K+h}V \tag{7.2.2}$$

$\xrightarrow{h=1 \text{ in (7.2.2)}}$

$$ {}_KV + \Pi_K = v\left(C_{K+1} q_{x+K} + {}_{K+1}V \cdot p_{x+K}\right)$$

$$\Longleftrightarrow \quad {}_KV + \Pi_K = v\left({}_{K+1}V + \left(C_{K+1} - {}_{K+1}V\right) q_{x+K}\right) \tag{7.2.3}$$

$\longrightarrow$ ${}_KV$ and Π_K are used to cover the PV (at $t=K$) of ${}_{K+1}V$ and the amount $C_{K+1} - {}_{K+1}V$ (needed at the end of the year), if the insured dies

$\longrightarrow$ $C_{K+1} - {}_{K+1}V$ net amount at risk

Further (7.2.3) admits the following decomposition of Π_K:

$$\Pi_K = \Pi_K^S + \Pi_K^R,$$

where $\Pi_K^S = {}_{K+1}V \cdot v - {}_KV$ savings premium (used to increase the net. prem. reserves)

and $\Pi_K^R = \left(C_{K+1} - {}_{K+1}V\right) \cdot v \cdot q_{x+K}$ risk premium (used to cover the net amount at risk)

Another reformulation of (7.2.3) gives

Th. 7.2.2 (discrete-time version of Thiele's diff. eq.)

$$\Pi_K + d \cdot {}_{K+1}V = \left({}_{K+1}V - {}_KV\right) + \Pi_K^R,$$

where $d := \frac{i}{1+i}$ is the discount rate, i.e. the int. rate credited at the beginning of each conversion period

$\longrightarrow$ prem. + int. on ${}_{K+1}V$ is used to cover the change of the net premium reserve and to finance the risk premium

7.3 Allocation of the total loss to policy years

Consider the whole life insurance of Section 7.2.
The <u>total loss of the insurer during the year $K+1$</u> can be defined as

$$\mathcal{L}_K = \begin{cases} 0, & \text{if } K \le K-1 \\ C_{K+1} \cdot v - ({}_K V + \Pi_K), & \text{if } K = K \\ {}_{K+1}V \cdot v - ({}_K V + \Pi_K), & \text{if } K \ge K+1 \end{cases} \qquad (7.3.1)$$

PV of C_{K+1} at $t=K$

PV of ${}_{K+1}V$ at $t=K$

$\Pi_K = \Pi_K^S + \Pi_K^T$

$$\Longleftrightarrow \quad \mathcal{L}_K = \begin{cases} 0 & \text{if } K \le K-1 \\ -\Pi_K^T + (C_{K+1} - {}_{K+1}V) \cdot v & \text{if } K = K \\ -\Pi_K^T & \text{if } K \ge K+1 \end{cases} \qquad (7.3.2)$$

def. of $\mathcal{L}_K$

$$\Longrightarrow \quad L = \sum_{K \ge 0} \mathcal{L}_K v^K \qquad (7.3.3)$$

$\longrightarrow$ decomposition of the total loss into losses during the corresponding policy years

$\longrightarrow$ <u>Th. 7.3.1</u> (Hallendorf's Theorem)

$$(\text{or } E[\mathcal{L}_K, \mathcal{L}_j] = 0, \; K \neq j$$

and

$$\text{Var}[L] = \sum_{K \ge 0} v^{2K} \text{Var}[\mathcal{L}_K]$$

<u>Proof</u>: $E[\mathcal{L}_K | K \ge K] \overset{(7.3.2)}{=} -\Pi_K^T q_{x+K} + \Pi_K^T - \Pi_K^T p_{x+K} = 0 \qquad (*)$

$\Rightarrow E[\mathcal{L}_K] = E[\mathcal{L}_K | K \ge K] \Pr(K \ge K) = 0$

$\Rightarrow (\text{or } E[\mathcal{L}_K, \mathcal{L}_j] = E[\mathcal{L}_K \cdot \mathcal{L}_j] = E[\mathcal{L}_K \mathcal{L}_j | K \ge j] \Pr(K \ge j)$

$\overset{(7.3.2)}{=} -\Pi_K^T \underbrace{E[\mathcal{L}_j | K \ge j]}_{\overset{(*)}{=} 0} \cdot \Pr(K \ge j) = 0$

$\Rightarrow \mathcal{L}_K, \mathcal{L}_j$ uncorrelated for $K < j \Rightarrow$ proof.