(38) (ii) Let N be a renewal proc. with i.i.d $W_i \sim Exp(\lambda)$
Then N is a hom. Poisson proc. with intensity $\lambda > 0$.

Proof: (i) It's sufficient to show that

$$P(T_1 \le x_1, \dots, T_n \le x_n) \overset{=: J_1}{} = P(W_1 \le x_1, \dots, W_1 + \dots + W_n \le x_n) \overset{=: J_2}{} \quad (*)$$

for all $x_i, i = 1, \dots, n, n \in \mathbb{N}$ for i.i.d $W_i \sim Exp(\lambda)$

$0 \le T_1 \le T_2 \le \dots$

$\longrightarrow$ w.l.o.g $0 \le x_1 \le x_2 \le \dots \le x_n$ in $(*)$

Recall: $F: \mathbb{R}^n \to \mathbb{R}$, $g(x_1, \dots, x_n)$ density of $(X_1, \dots, X_n)$, i.e.

$$P(X_1 \le x_1, \dots, X_n \le x_n) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} g(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

Then

$$E[F(X_1, \dots, X_n)] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(x_1, \dots, x_n) \cdot g(x_1, \dots, x_n)\, dx_1 \dots dx_n \quad (**)$$

Consider $n = 2$

$$J_2 = E[1_{\{0 \le W_1 \le x_1, 0 \le W_1 + W_2 \le x_2\}}(W_1, W_2)]$$

$$F(w_1, w_2) := 1_{\{0 \le w_1 \le x_1, 0 \le w_1 + w_2 \le x_2\}}(w_1, w_2), \quad g(w_1, w_2) = \lambda e^{-\lambda w_1} \lambda e^{-\lambda w_2}$$

$$\overset{(**)}{\Longrightarrow} J_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(w_1, w_2) \cdot g(w_1, w_2)\, dw_1\, dw_2$$

On the other hand use

$$\{T_1 \le x_1, T_2 \le x_2\} = \{N(x_1) \ge 1, N(x_2) \ge 2\}, \text{Rem. 9.1.2(ii)}, (**)$$

to verify that $J_1 = J_2$ (See Exerc. 6) $\Longrightarrow$ proof

(ii) See Mikosch

Rem. 9.1.3.2.:

(i) Since

$$S(t) = \sum_{i \ge 1} X_i 1_{[0,t]}(T_i), \quad t \ge 0 \quad (\text{see } (8.1))$$

$$T_i = W_1 + \dots + W_i$$

one can employ the indep. i.i.d. sequences (X_i) and (W_i) to simulate $S(t)$ (See later)

(ii) Since $W_i, i \ge 1$ possesses a density (see 9.3.1.1) we see that

$\Longrightarrow$ $W_i > 0$, $i \ge 1$ with prob. 1

$0 < T_1 < T_2 < \dots$ with prob. 1

$\Longrightarrow$ $N(t)$ only has jumps of order 1 with prob. 1

What is the distr. of $(T_1,\dots,T_n)$ and $(W_1,\dots,W_n)$?

→ **Prop. 9.1.3.3** : Let N be a (inhom.) Poisson proc. with $\mu(t) = \int_0^t \lambda(s)\,ds$, $\lambda(x) > 0$ continuous. Then

(i) the density of $(T_1,\dots,T_n)$ is given by

$$f_{T_1,\dots,T_n}(x_1,\dots,x_n) = e^{-\mu(x_n)} \prod_{i=1}^n \lambda(x_i)\, 1_{\{0<x_1<\dots<x_n\}}$$

(ii) the density of $(W_1,\dots,W_n)$ takes the form

$$f_{W_1,\dots,W_n}(x_1,\dots,x_n) = e^{-\mu(x_1+\dots+x_n)} \prod_{i=1}^n \lambda(x_1+\dots+x_i)\, 1_{x_i>0}$$

Proof : (consider e.g. the case $\lambda(x) \equiv \lambda > 0$)

(i) Use $(T_1,\dots,T_n) = (W_1, \dots, \sum_{i=1}^n W_i)$ and the property that $W_i \sim \mathrm{Exp}(\lambda)$ i.i.d.

(ii) $f_{W_1,\dots,W_n} = \prod_{i=1}^n f_{W_i}$

⇒ proof

Now let us discuss an important property of the Poisson proc. which can be e.g. applied to determine the distr. of the generalized total claim amount

$$S(t) = \sum_{i=1}^{N(t)} g(T_i, X_i),$$

where $g : \mathbb{R}^2 \to \mathbb{R}$

→ ## 9.1.4 The order statistics property of the Poisson process

Notation : Let $X_1,\dots,X_n$ be indep. r.v.'s

→ $X_{(j)}$ jth smallest of $X_1,\dots,X_n$

→ r.v.'s $X_{(1)} \le \dots \le X_{(n)}$ order statistics of $X_1,\dots,X_n$

Ex. : $X_{(1)} = \min(X_1,\dots,X_n)$, $X_{(n)} = \max(X_1,\dots,X_n)$

We are coming to a main result of our course :

Th. 9.1.4.1 (Order statistics property of N)

Let N be a Poisson proc. with $\mu(t) = \int_0^t \lambda(s)\,ds$, $\lambda(x) > 0$ continuous and $0 < T_1 < T_2 < \dots$ with prob. 1

Then

$$P(T_1 \le x_1,\dots,T_n \le x_n \mid N(t)=n) = P(X_{(1)} \le x_1,\dots,X_{(n)} \le x_n)$$

for $X_1,\dots,X_n$ i.i.d r.v.'s with density $f(x) = \frac{\lambda(x)}{\mu(t)}$, $0 < x \le t$

Moreover

$$P(T_1 \le x_1,\dots,T_n \le x_n \mid N(t)=n) = \int_0^{x_1}\cdots\int_0^{x_n} f_{T_1,\dots,T_n}(y_1,\dots,y_n \mid N(t)=n)\,dy_1\cdots dy_n$$

where $f_{T_1,\dots,T_n}$ ↖ cond. density

$$f_{T_1,\dots,T_n}(x_1,\dots,x_n \mid N(t)=n) = f_{(X_{(1)},\dots,X_{(n)})}(x_1,\dots,x_n) = \begin{cases} \frac{n!}{(\mu(t))^n} \prod_{i=1}^n \lambda(x_i), & 0 < x_1 < \dots < x_n \le t \\ 0 & \text{else} \end{cases}$$

(40) Recall: $P(A|B) \overset{def}{=} \frac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$, A, B events

Proof: Let us show that

$$\lim_{h_i \downarrow 0, i=1,\dots,n} \frac{P(T_1 \in (x_1, x_1+h_1], \dots, T_n \in (x_n, x_n+h_n] \mid N(t)=n)}{h_1 \cdots h_n}$$

$$= \frac{n!}{(\mu(t))^n} \prod_{i=1}^{n} \lambda(x_i), \quad 0 < x_1 < \dots < x_n < t \qquad (*)$$

Choose $h_i, i=1,\dots,n$ small enough s.t.
$(x_i, x_i+h_i], i=1,\dots,n$
are disjoint

Consider the case $n=2$.
Then

$$\{T_1 \in (x_1, x_1+h_1],\ T_2 \in (x_2, x_2+h_2],\ N(t)=2\}$$
$$= \{N(0, x_1]=0,\ N(x_1, x_1+h_1]=1,\ N(x_1+h_1, x_2]=0,$$
$$N(x_2, x_2+h_2]=1,\ N(x_2+h_2, t]=0\} \qquad (**)$$

Recall: $N(a,b]$ = number of claims between a and b

$\overset{(**)}{\Longrightarrow}$ $P(T_1 \in (x_1, x_1+h_1],\ T_2 \in (x_2, x_2+h_2],\ N(t)=2)$
$= P(N(0, x_1]=0,\ N(x_1, x_1+h_1]=1,\ N(x_1+h_1, x_2]=0,$
$N(x_2, x_2+h_2]=1,\ N(x_2+h_2, t]=0)$

$\overset{\text{indep. increm. of } N}{=}$ $P(N(0, x_1]=0)\, P(N(x_1, x_1+h_1]=1) \cdot P(N(x_1+h_1, x_2]=0)$
$\cdot P(N(x_2, x_2+h_2]=1) \cdot P(N(x_2+h_2, t]=0)$

$\overset{\text{increm. Poisson distr}}{=}$ $e^{-\mu(x_1)} \cdot e^{-\mu(x_1, x_1+h_1]}\, \mu(x_1, x_1+h_1]\, e^{-\mu(x_1+h_1, x_2]}$
$\cdot e^{-\mu(x_2, x_2+h_2]}\, \mu(x_2, x_2+h_2] \cdot e^{-\mu(x_2+h_2, t]}$

$$= e^{-\mu(t)}\, \mu(x_1, x_1+h_1) \cdot \mu(x_2, x_2+h_2]$$

$$\Rightarrow P(T_1 \in (x_1, x_1+h_1],\ T_2 \in (x_2, x_2+h_2] \mid N(t)=2) / h_1 \cdot h_2$$
$$= P(T_1 \in (x_1, x_1+h_1],\ T_2 \in (x_2, x_2+h_2],\ N(t)=2) / h_1 \cdot h_2 \cdot P(N(t)=2)$$
$$= \frac{2!}{(\mu(t))^2} \cdot \frac{\mu(x_1, x_1+h_1]}{h_1} \cdot \frac{\mu(x_2, x_2+h_2]}{h_2}$$
($\frac{\mu(x_1, x_1+h_1]}{h_1} \to \lambda(x_1)$, $\frac{\mu(x_2, x_2+h_2]}{h_2} \to \lambda(x_2)$)

$$\xrightarrow{h_1, h_2 \downarrow 0} \frac{2!}{(\mu(t))^2} \lambda(x_1)\lambda(x_2) = (*) \text{ for } n=2$$

Similarly (*) can be established for $(x_i, x_i+h_i], i=1,\dots,n$

$\Rightarrow P(T_1 \le x_1, \dots, T_n \le x_n \mid N(t)=n)$ has a density given by (*)

The fact that

$$f_{X_{(1)},\dots,X_{(n)}} = (*)$$

follows from the next Lemma $\Rightarrow$ proof

Lemma 9.1.4.2 (Joint density of order statistics)
Let $X_i, i \ge 1$ be i.i.d with density f. Then

$$f_{X_{(1)},\dots,X_{(n)}}(x_1,\dots,x_n) = n! \prod_{i=1}^{n} f(x_i) \cdot 1_{\{x_1 < \dots < x_n\}}$$

Proof: See e.g. Mikosch, p. 28

Ex. 9.1.4.3 (Order statistics property of hom. N)

hom. N $\longrightarrow$ $\mu(t) = \lambda t$, $\lambda > 0$ (Th. 9.1.4.1 $\Rightarrow$)

$$f_{T_1,\dots,T_n}(x_1,\dots,x_n \mid N(t)=n) = n!\, t^{-n}, \quad 0 < x_1 < \dots < x_n < t$$

Let $U_1, \dots, U_n$ be i.i.d with $U_i \sim U(0,t)$, that is

$$P(U_i \le x) = \frac{x}{t}, \quad x \in [0,t]$$

(L. 9.1.4.2 $\Rightarrow$)

$$f_{U_{(1)},\dots,U_{(n)}}(x_1,\dots,x_n) = f_{T_1,\dots,T_n}(x_1,\dots,x_n \mid N(t)=n)$$

(indep. of λ!)

Ex. 9.1.4.4 (applied to symmetric functions)
Function g on $\mathbb{R}^n$ symmetric $\iff g(x_1,\dots,x_n) = g(x_{\pi(1)},\dots,x_{\pi(n)})$ for all permutations π of $\{1,\dots,n\}$

Ex.: $g = \prod_{i=1}^{n} x_i$ or $g = \sum_{i=1}^{n} x_i$ (Th. 9.1.4.1 $\Rightarrow$)

$$\big(g(T_1,\dots,T_n) \mid N(t)=n\big) \overset{d}{=} g(X_{(1)},\dots,X_{(n)}) = g(X_1,\dots,X_n)$$

(distr. of $g(T_1,\dots,T_n)$ given $N(t)=n$)

Using Th. 9.1.4.1 and the last two examples give (see Mikosch)

Prop. 9.1.4.5 (Distr. of a generalized total claim amount)
Let N be a hom. Poisson proc. with $\lambda > 0$.
Then for all $g: \mathbb{R}^2 \to \mathbb{R}$:

$$\tilde{S}(t) := \sum_{i=1}^{N(t)} g(T_i, X_i) \overset{d}{=} \sum_{i=1}^{N(t)} g(t \cdot U_i, X_i),$$

where
(U_i) i.i.d $U(0,1)$ indep. of (X_i) and (T_i) (indep.)

Ex. 9.1.4.6 : Present value of the claim payments in $S(t)$ is given by $\tilde{S}(t) := \sum_{i=1}^{N(t)} e^{-rT_i} X_i$

$\longrightarrow$ $g(T,X) = e^{-rT}X$

Ex. 9.1.4.7 : Modelling arrivals of incurred but not reported claims:
reality: claim i is rather reported at $T_i + V_i$ than at T_i with delay V_i

$\longrightarrow$ # of claims reported up to time t is given by
$\tilde{N}(t) := \sum_{i=1}^{N(t)} 1_{[0,t]}(T_i + V_i)$
Choose X_i as V_i in Prop. 9.1.4.5

9.2 N as a renewal process

Recall Def. 9.1.3.1:

$W_i \geq 0$, $i \geq 1$ i.i.d, $T_0 = 0$, $T_n = W_1 + \dots + W_n$, $n \geq 1$

$\longrightarrow$ $N(t) = \#\{i \geq 1 : T_i \leq t\}$, $t \geq 0$

renewal process

$\longrightarrow$ generalization of the hom. Poisson proc. with i.i.d $W_i \sim \text{Exp}(\lambda)$

Rem.: The name "renewal process" comes from applications to model the number of renewals $N(t)$ of a technical device (e.g. lightbulbs in a factory) up to time t.

Why a renewal process instead of a hom. Poisson proc.?

$\longrightarrow$ It's characteristic e.g. for windstorm insurance that inter-arrival may become very large

$\longrightarrow$ modelling of such W_i not captured by $\text{Exp}(\lambda)$

$\longrightarrow$ distributions with heavier tails (like e.g. the log-normal or Pareto distr. are more appropriate

Recall: Pareto distr. $\Leftrightarrow$ density f given by

$$f(x) = \begin{cases} 0, & x \leq x_0 \\ \frac{a}{x_0}\left(\frac{x_0}{x}\right)^{a+1}, & x > x_0 \end{cases}$$

for x_0 and $a > 0$ constants

We shall discuss asymptotic properties of the renewal proc. $N(t)$, which play a central role in insurance business!

Theorem 9.2.1 (SLLN for the renewal proc.)

Let W_i, $i \geq 1$ be the inter-arrival times of a renewal proc. $N(t)$ with $\mathbb{E}[W_1] = \frac{1}{\lambda}$, $\lambda > 0$.

Then the SLLN of $N(t)$ holds in the following sense:

$$\frac{N(t)}{t} \xrightarrow[t\to\infty]{} \lambda$$

Proof: Idea: SLLN applied to (W_i)

By def. of N we have that

$$N(t)=n \iff T_n \le t < T_{n+1} \quad , n\in\mathbb{N}_0$$

$$\Rightarrow \quad T_{N(t)} \le t < T_{N(t)+1}$$

$\Rightarrow$ "sandwich inequality":

$$\frac{T_{N(t)}}{N(t)} \le \frac{t}{N(t)} \le \frac{T_{N(t)+1}}{N(t)+1}\,\frac{N(t)+1}{N(t)} \qquad (*)$$

On the other hand the SLLN gives

$$\frac{1}{n}T_n = \frac{1}{n}\sum_{i=1}^{n} W_i \overset{\text{i.i.d}}{\longrightarrow} E[W_1] = \frac{1}{\lambda}$$

$$\Rightarrow \quad T_n \to \infty \quad \text{for } n\to\infty$$

$$\Rightarrow \quad N(t) \to \infty \quad \text{for } t\to\infty$$

$$\Rightarrow \quad \frac{T_{N(t)}}{N(t)} \,,\, \frac{T_{N(t)+1}}{N(t)+1} \xrightarrow[t\to\infty]{} E[W_1] = \frac{1}{\lambda}$$

$\overset{(*)}{\Rightarrow}$ $\quad \frac{t}{N(t)} \to \frac{1}{\lambda} \quad \Rightarrow$ proof.

We shall study the mean behaviour of $E[N(t)]$.

$\rightarrow$ **Theorem 9.2.2** (Elementary renewal theorem)

If $E[W_1] = \frac{1}{\lambda}$ for $\lambda > 0$ then

$$\frac{E[N(t)]}{t} \xrightarrow[t\to\infty]{} \lambda$$

Proof: Idea: Wald's identity

$$W_i^{(b)} := \min(W_i, b) \,,\; T_i^{(b)} := W_1^{(b)} + \ldots + W_i^{(b)} \,,\; i\ge 1 \,,\; b>0$$

$\rightarrow$ truncated W_i, T_i

$$N_b(t) := \#\{i\ge 1 : T_i^{(b)} \le t\} \,,\; t\ge 0$$

Def. 9.1.3.1 $\Rightarrow$ $N_b(t)$ renewal proc.

$$T_n \ge T_n^{(b)} \Rightarrow N_b(t) \ge N(t)$$

$$\Rightarrow \quad \varlimsup_{t\to\infty} \frac{E[N(t)]}{t} \le \varlimsup_{t\to\infty} \frac{E[N_b(t)]}{t} \qquad (+)$$

Now use Wald's identity to obtain that

$$E\left[T^{(b)}_{N_b(t)+1}\right] = E[N_b(t)+1]\cdot E[W_1^{(b)}] \qquad (++)$$

(45) By def. we have that

$$T^{(b)}_{N_b(t)} = W_1^{(b)} + \dots + W^{(b)}_{N_b(t)} \leq t$$

$$\Rightarrow \quad T^{(b)}_{N_b(t)+1} = T^{(b)}_{N_b(t)} + W^{(b)}_{N_b(t)+1} \leq t+b \qquad (+++)$$

$$\Rightarrow \quad \overline{\lim_{t\to\infty}} \frac{E[N(t)]}{t} \overset{(+),(++)}{\leq} \overline{\lim_{t\to\infty}} \frac{E[T^{(b)}_{N_b(t)+1}]}{t\,E[W_1^{(b)}]} \overset{(+++)}{\leq}$$

$$\overline{\lim_{t\to\infty}} \frac{t+b}{t\,E[W_1^{(b)}]} = \frac{1}{E[W_1^{(b)}]} \xrightarrow[b\to\infty]{} \frac{1}{E[W_1]} = \lambda$$

$W_1^{(b)} \searrow W_1$ for $b \to \infty$

One also shows that

$$\underline{\lim_{t\to\infty}} \frac{E[N(t)]}{t} \geqslant \lambda$$

$$\Rightarrow \quad \lim_{t\to\infty} \frac{E[N(t)]}{t} = \lambda \quad \Rightarrow \text{proof}$$